An Analysis Sketchbook

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Contents

Preface

This book grew out of our desire to teach a first-semester advanced calculus course that would not only engage our students, but that would allow them to discover for themselves the "big picture" ideas that form the foundation of much of the study of analysis. We wanted our students to investigate examples, ask questions, make conjectures, and prove theorems. We wanted them to see analysis as a beautiful picture to be painted in broad strokes, and not as simply an exercise in color-by-numbers pointillism. And, of course, we wanted them to see why this picture needed to be painted in the first place.

Our approach is marked by three distinguishing characteristics. First, this book is designed not simply to be read, but to be engaged and experienced in a hands-on, interactive manner. The main body of material is developed through a sequence of 14 activities, each of which is punctuated by questions that must be answered by the reader before moving on to subsequent material. In this way, our approach is similar to that of the Moore method. In fact, when we use these activities in our own classrooms, we spend most of our time in class having students present their solutions to the in-text questions and critique the solutions of others.

Second, we have adopted a "less is more" philosophy in choosing which topics and results to include. There is only so much material that can be covered well in an introductory analysis course, and we have made no effort to be comprehensive. Instead, our goal has been to develop in a deep and meaningful way the concepts, techniques, and ways of thinking that are central to the study of analysis. Although we have chosen to focus on big ideas, we have done so in a rigorous way. As such, students who wish to take further courses in analysis will be well-prepared to do so.

The final distinguishing characteristic of this text is its approach to the real numbers. The very first activity exposes the incompleteness of the rational numbers, and the next five activities develop both the intuition and the formal machinery necessary to construct the reals and prove their completeness. Until we have completed the construction in Activity 7, we make no explicit reference to irrational numbers such as $\sqrt{2}$. Furthermore, we use rational values of ε in our definitions of convergent and Cauchy sequences, noting only later in the text that these definitions are in fact equivalent to the more standard ones (which allow ε to be any positive real number). We attempt to demonstrate a need for the real numbers before we construct or use them, and our experience is that this approach is quite effective. After several weeks of not being able to mention $\sqrt{2}$, π, e, or any other familiar real numbers, students feel (at the very least) relieved once they are finally able to do so. They also tend to be surprised by the amount of effort it takes to define the real numbers and prove properties of the reals that they have taken for granted in the past.

As mentioned earlier, we have used the 14 activities in this text as the basis of a student-driven, Moore-style course in advanced calculus. We have found that, within this framework, there is more than enough material for a typical 15 week semester. However, for those who wish to cover additional topics, we have also developed several supplemental projects that build upon and extend the material from the main body of the text. These activities are available online at `www.somewebaddress.com`.

If possible, the activities in the text should be covered in order, although many of the more detailed proofs can be safely omitted if need be. Also, with only a few exceptions, the exercises are optional and not typically referenced later in the text.

Finally, although we have used this text in lieu of a more standard textbook, we anticipate that it could also be used as a supplemental text, or as a text for independent study. And, since our experiences with this material have been primarily within small, Moore-style classes, we would be curious to know how others use these activities with different class sizes and formats. We welcome these and any other comments, questions or suggestions for improvement.

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Activity 1

Calculus in $\mathbb{Q}$?

Focus Questions

- What is Newton's method? How can Newton's method be used to find roots of functions?
- Why are the rational numbers inadequate for the problem of finding roots of polynomials?

Introduction

One of the most basic and yet important applications of calculus is that of finding roots of functions. In this activity, we will see what happens when we try to use Newton's method to find a root of a polynomial function, all while restricting our universe of numbers to just the rationals. Our investigations will give us one example of why the real numbers are essential to the study of calculus.

Our main objective in this activity is to find an important property of the real numbers that is not shared by the rationals. To do so, we will assume throughout the activity that **the only numbers that exist are rational numbers** (and subsets of rationals, such as the integers). Seeing how this assumption restricts us will ultimately demonstrate our need for the real numbers and will also suggest one way to formally define the reals. For those concerned about historical precedent, be assured that this little experiment is nothing more than a journey back to about 300 B.C., a time when greek mathematicians such as Euclid also denied the existence of any numbers beyond the rationals.

A Quick Review of Newton's Method

You may remember Newton's method from your first-semester calculus course. The idea behind it is fairly simple: the method uses sequences of tangent line approximations of a function to approximate the function's roots. It is not too difficult to show that, given a function f and a point a in the domain of f, the x-intercept of the line tangent to f at the point a is exactly equal to

$$a - \frac{f(a)}{f'(a)}.$$

Thus, given an initial approximation x_0 of a root of f, Newton's method uses the x-intercept of the line tangent to f at x_0 to find a potentially better approximation, say x_1, of the root. The process is then repeated, but this time using x_1 as the starting point. Continuing in this fashion produces the sequence of approximations given by the following recurrence relation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In most cases (but not all), this sequence of approximations, or *iterates*, gets closer and closer to the exact value of the desired root. Figure 1 gives a graphical illustration of how Newton's method works, and Exercise 3 at the end of the activity provides examples of how the method can fail.

Newton's Method in $\mathbb{Q}$

Now let's consider a specific example. We'll begin with the following polynomial:

$$p(x) = \frac{3}{4}x^2 - \frac{3}{2}.$$

Question 1.1.

(a) Apply one iteration of Newton's method for $p(x)$ with $x_0 = 3/2$. What is the result of the first iteration? (That is, what is x_1?)

(b) Apply another iteration of Newton's method for $p(x)$. What the result of this iteration? (That is, what is x_2?)

(c) Show that applying two iterations of Newton's method for $p(x)$, starting with any non-zero rational number $x_0 = a$, yields (for the second iterate):

$$x_2 = \frac{a^4 + 12a^2 + 4}{4a^3 + 8a}.$$

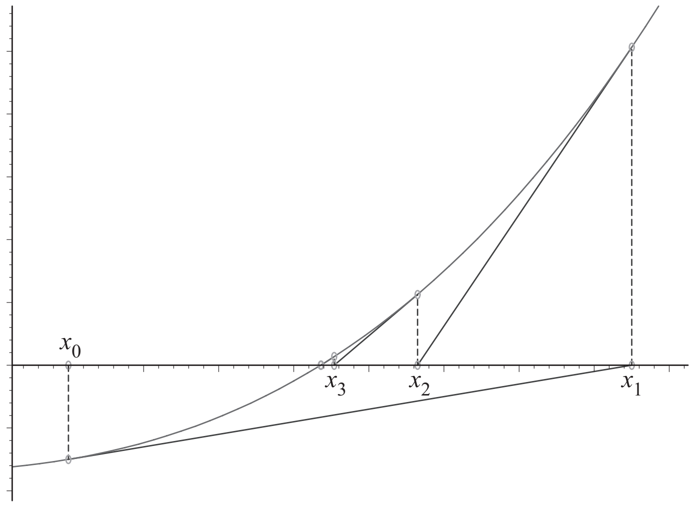

Figure 1.1: An illustration of Newton's method

Question 1.2. Prove that if you apply an iteration of Newton's method for $p(x)$ to a non-zero rational number x_n, then x_{n+1} will also be a non-zero rational number. Deduce that the sequence of iterates starting with $x_0 = 3/2$ are all non-zero.

Question 1.3. Prove that if you apply an iteration of Newton's method for $p(x)$ to a rational number x_n such that $1 < x_n < 2$, then $1 < x_{n+1} < 2$. Deduce that the sequence of iterates starting with $x_0 = 3/2$ are all between 1 and 2.

Question 1.4. Prove that if you apply an iteration of Newton's method for $p(x)$ to any non-zero rational number x_n, then $x_{n+1}^2 > 2$. What can you deduce about the sequence of iterates starting with $x_0 = 3/2$?

Question 1.5. Prove that if you apply an iteration of Newton's method for $p(x)$ to a positive rational number x_n with $x_n^2 > 2$, then $x_{n+1} < x_n$. Deduce that in the sequence of iterates starting with $x_0 = 3/2$, every iterate is smaller than the previous one.

Question 1.6. Prove that if you apply an iteration of Newton's method for $p(x)$ to a rational number $x_n > 1$, then

$$\left|x_{n+1}^2 - 2\right| < \frac{1}{2}\left|x_n^2 - 2\right|.$$

What can you deduce about the sequence of iterates starting with $x_0 = 3/2$?

Question 1.7. Prove that if you apply two iterations of Newton's method for $p(x)$ to a positive rational number x_n with $x_n^2 > 2$, then

$$|x_{n+1} - x_{n+2}| < \frac{1}{2}|x_n - x_{n+1}|.$$

What can you deduce about the sequence of iterates starting with $x_0 = 3/2$?

Question 1.8. In light of your answers to Questions 1.2 through 1.7, what are the iterates of Newton's method for $p(x)$ (starting with $x_0 = 3/2$) approaching? In other words, what root of $p(x)$ will Newton's method find? (Note: Remember that the only numbers that exist are the rationals!)

Exercises

(1) Prove that there does not exist a rational number whose square is 2. (Hint: Reason by contradiction, using the fact that every rational number can be written in lowest terms.)

(2) Prove that the x-intercept of the line tangent to $f(x)$ at the point x_n is exactly

$$x_n - \frac{f(x_n)}{f'(x_n)}.$$

(3) For each of the following functions, describe in a precise way what happens when you try to find a root of the function by applying Newton's method with the given initial point. Comment on any interesting behavior you observe, and explain why this behavior occurs.

(a) $f(x) = 7x^4 - 57x^2 + 108$; $x_0 = 2$

(b) $f(x) = x^3 - 6x^2 + 7x + 2$; $x_0 = 1$

(c) $f(x) = x^2 + 1$; $x_0 = 0.5$

(d) $f(x) = 3\cos(x) - 2$; $x_0 = 0.01$

(4) Consider the sequence $\boldsymbol{x}$ defined by

$$x_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

for all $n \geq 0$.

(a) Explain why x_n is a rational number for each n.

(b) Prove that $\boldsymbol{x}$ does not converge to a rational number. (Hint: Suppose that $\boldsymbol{x}$ does converge to a rational number, say $l = p/q$, where p and q are integers and $q \neq 0$. Argue that $x_{2n-1} < l < x_{2n}$ for all $n \geq 1$, and use this fact arrive at a contradiction by proving that, for all $n \geq 1$, q does not divide $(2n)!$.)

(5) Assume for the purposes of this question that the only numbers that exist are those of the form $a + b\sqrt{2}$, where a and b are rational numbers. Let $f(x) = x^4 - 5x^2 + 6$. Describe what would happen if Newton's method were used to find a root of $f(x)$ given each of the following values of x_0:

(a) $x_0 = 0$

(b) $x_0 = 1$

(c) $x_0 = 2$

Activity 2

How Close is "Close Enough?"

Focus Questions

- Intuitively, what does it mean for a sequence of numbers to converge?
- Intuitively, what does it mean for a sequence of numbers to accumulate?

Introduction

In Activity 1, we investigated the long-term behavior of a sequence of rational numbers defined by applying Newton's method to a particular polynomial function. The elements of this sequence seemed to be getting closer and closer to some number, but not one that existed in our universe of just the rationals.

When dealing with sequences of numbers, we often use phrases like *closer and closer*, *approaching*, *eventually*, *converges to*, and so on. We may have an intuitive sense of what these terms mean, but in order to study the behavior of sequences in a mathematically precise and meaningful way, we are going to need to move toward a more formal framework. We will begin to do so in this activity by informally defining the notions of convergence and accumulation. In subsequent activities, we will make these informal definitions much more precise.

Close to Something

Let's begin by considering the following sequence of numbers:

$$1, 4, \frac{5}{2}, \frac{13}{4}, \frac{23}{8}, \frac{49}{16}, \ldots. \tag{2.1}$$

Notice that after the first two numbers in the sequence, each subsequent number is the average of the previous two (so, for example, $\frac{23}{8} = \frac{5/2+13/4}{2}$). As is often the case when studying sequences, we would like to know the long-term behavior of this sequence of numbers.

Question 2.1. Calculate the next three numbers of the sequence given in Equation (2.1). Based on these calculations, what do you think the long-term behavior of the sequence will be?

It can be quite cumbersome to repeatedly use phrases like "the third element of the sequence that we wrote down in Equation (2.1)" to refer to the numbers in the particular sequences we are investigating. To simplify the way we discuss such sequences, we will adopt some very simple and natural notation. In particular, we will refer to the sequence that we wrote down in Equation (2.1) as sequence $\boldsymbol{a}$. We will then use subscript notation to refer to the individual elements of $\boldsymbol{a}$, so that, for example, $a_3 = \frac{5}{2}$. Obviously, we can (and will) use similar notation to refer to other sequences we encounter throughout our investigations.

Question 2.2. It can be shown that for some rational numbers α and β,

$$a_n = \alpha + \beta \left(\frac{-1}{2}\right)^n$$

for each positive integer n. Find α and β.

Question 2.3. In light of your answer to Question 2.2, what do you think the long-term behavior of sequence $\boldsymbol{a}$ will be, and why? Is there a number p such that, eventually, the numbers a_n are as close to p as one could possibly want them to be?

Question 2.3 suggests the following informal definition of convergence:

Informal Definition 2.1. A sequence $\boldsymbol{s}$ is said to *converge* to a number l provided that **eventually** the elements of $\boldsymbol{s}$ (that is, s_1, s_2, s_3, ...) become **as close to l as we want them to be.** When we say that $\boldsymbol{s}$ *converges* (without specifying a value of l), we mean that there exists some number l such that $\boldsymbol{s}$ converges to l.

The idea behind Informal Definition 2.1 is this: when we say that $\boldsymbol{s}$ converges to l, we mean that no matter how close we want the elements of $\boldsymbol{s}$ (that is, the s_n) to be to l, they will eventually be that close. So, for instance, if a sequence $\boldsymbol{s}$ converges to 2, then eventually (once we move beyond a certain point in the sequence) the elements of $\boldsymbol{s}$ will all be very close to 2, say in between 1.99 and 2.01. Not close enough? Well, eventually (if we look even further into the sequence), all of the elements of $\boldsymbol{s}$ will be in between 1.999 and 2.001. Still not close enough? No worries – we can repeat this process for as long as you would like. In fact, the central idea behind saying that $\boldsymbol{s}$ converges to 2 is that no matter how close we say is "close enough," we will be able to find some point in $\boldsymbol{s}$ such that beyond that point, all of the elements of $\boldsymbol{s}$ are "close enough" to 2.

Question 2.4. Which of the following sequences converge, and which do not? For each sequence that does converge, find the number l to which the sequence converges, and use Informal Definition 2.1 to explain why the sequence does in fact converge to l. For each sequence that does not converge, use Informal Definition 2.1 to explain why no such l exists.

(a) The sequence $\boldsymbol{w}$ defined by

$$w_n = \frac{(-1)^n}{n}$$

for each positive integer n.

(b) The sequence $\boldsymbol{x}$ defined by

$$x_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 7 + 1/n & \text{if } n \text{ is even} \end{cases}$$

for each positive integer n.

(c) The sequence $\boldsymbol{y}$ defined by

$$y_n = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 3 - 1/n & \text{if } n \text{ is even} \end{cases}$$

for each positive integer n.

(d) Let $\boldsymbol{z}$ be the sequence defined by

$$z_n = \begin{cases} 3 + 1/n & \text{if } n \text{ is not a power of 10} \\ \frac{14}{5} - 1/n & \text{if } n \text{ is a power of 10} \end{cases}$$

for each positive integer n.

Close To Something vs. Close Together

In the first part of this activity, we looked at what it meant for a sequence to converge. However, we ignored the problem that showed up in the rational numbers:

in order to converge, a sequence has to get close to *something*. But what if there isn't something to get close to?

Let's look back at the sequence $\boldsymbol{a}$ from the beginning of the activity. Recall that we found rational numbers α and β such that

$$a_n = \alpha + \beta \left(\frac{-1}{2}\right)^n$$

for each positive integer n.

Using this definition of $\boldsymbol{a}$, let's again consider the long-term behavior of the sequence. This time, however, let's do so without referencing any numbers except those that occur as elements of $\boldsymbol{a}$. Informal Definition 2.2 suggests one way to do so.

Informal Definition 2.2. A sequence $\boldsymbol{s}$ is said to *accumulate* provided that **eventually** the elements of $\boldsymbol{s}$ become **as close to each other as we want them to be.**

Question 2.5.

(a) How is Informal Definition 2.2 similar to Informal Definition 2.1? How are the two definitions different?

(b) Does sequence $\boldsymbol{a}$ accumulate? Why or why not?

(c) Which (if any) of the other sequences from this activity accumulate? Justify each of your answers using Informal Definition 2.2.

(d) Does the sequence of numbers arising from Newton's method in Activity 1 accumulate? Why or why not?

Exercises

(1) Characterize all of the values of α, β, x, and y for which the sequence $\boldsymbol{s}$ defined by

$$s_n = \alpha x^n + \beta y^n$$

converges.

(2) Let $\boldsymbol{S}$ be the sequence of partial sums defined by

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

for each positive integer n. Does $\boldsymbol{S}$ accumulate? Why or why not?

(3) Let $\boldsymbol{S}$ be the sequence of partial sums defined by

$$S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$$

for each positive integer n. Does $\boldsymbol{S}$ accumulate? Why or why not?

(4) Let $\boldsymbol{s}$ be a sequence of rational numbers, and let S_n be the sequence of partial sums defined by

$$S_n = \sum_{k=1}^{n} s_k$$

for each positive integer n.

(a) If $\boldsymbol{s}$ converges, must $\boldsymbol{S}$ also converge? Give a convincing argument or counterexample to justify your answer.

(b) If $\boldsymbol{S}$ converges, must $\boldsymbol{s}$ also converge? Give a convincing argument or counterexample to justify your answer.

(5) Does every convergent sequence of rational numbers accumulate? Why or why not?

(6) Under what circumstances does a sequence of integers converge? Be specific and precise.

(7) Suppose a sequence $\boldsymbol{s}$ satisfies the property that

$$|s_{n+1} - s_n| < \frac{1}{n}$$

for each positive integer n. Does $\boldsymbol{s}$ necessarily accumulate? Why or why not?

(8) Consider all sequences $\boldsymbol{s}$ that satisfy the following property:

> For each positive integer N, there exists m, $n > N$ such that $s_m < 0$ and $s_n > 0$.

Do such sequences always, sometimes, or never converge? Give a proof or a pair of examples (whichever is most appropriate) to justify your answer.

Activity 3

Close Enough: The Game

Introduction

In Activity 2, we discussed informally what it means for a sequence to converge to a particular number. In this activity, we will continue to explore the notion of convergence by playing a game that is based on the idea of being "close enough" to a desired limit. The complete rules of the game are stated below.

Game Rules

- There are two players. In the first round, Player 1 is the *chooser* and Player 2 is the *guesser*. After each round, the players change roles.
- To begin a round, the chooser selects a sequence (which, for clarity, we will refer to as s).
- After the chooser has selected a sequence, the guesser then determines a *target*. This target is the number that the guesser must try to make the elements of s get close to.
- The chooser picks a positive distance that determines exactly how close to the target is *close enough* (that is, how close to the target the guesser must make the elements of s). Elements of s that are less than this distance from the target are said to be *in the goal*.
- The guesser must determine how far to go in s so that all of the subsequent elements of s will be in the goal.

- If the guesser is able to find a place in s past which all the elements of s are in the goal, then the guesser wins the round. Otherwise, the chooser wins the round.

Analysis

With a partner, play several rounds of the game just described. Play the game enough times and with enough different sequences to be able to give clear and precise answers to each of the questions stated below.

Question 3.1. Are there optimal strategies for each player? If so, describe these optimal strategies in detail.

Question 3.2. Are there sequences for which either player could win depending on how they play? Either give an example of such a sequence, or explain why no such sequences exist.

Question 3.3. Are there sequences for which one of the players would be guaranteed to win, provided that they played correctly? Either give an example of such a sequence, or explain why no such sequences exist.

Question 3.4. Does the game favor the chooser, the guesser, or neither? In other words, if both players played optimal strategies, who would be more likely to win?

Question 3.5. What could be done to make the game more fair and/or more interesting? Suggest at least one possible change.

Activity 4

Defining Convergence

Focus Questions

- What is the precise definition of convergence for a sequence of numbers?
- What are some strategies for proving or disproving the convergence of a given sequence?

Introduction

In Activity 2, we said that a sequence $\boldsymbol{s}$ converges to a number l exactly when the terms of $\boldsymbol{s}$ (which we called s_n) can eventually be made as close as we would like them to be to l. In this activity, we will use the the ideas from Activity 3 to make our definition of convergence more mathematically precise.

A Recap of *Close Enough: The Game*

In *Close Enough: The Game* (Activity 3), you probably observed that the chooser can always guarantee a win by simply choosing a sequence (in the first step of the game) that does not converge. On the other hand, if the chooser picks a convergent sequence, then the guesser should always be able to win, provided that he or she selects the correct target for the sequence (which we'll call l). But what is the correct target? What strategy should the guesser employ when the chooser picks $\boldsymbol{s}$ to be a convergent sequence?

To answer this question, recall that the guesser wins if and only if he or she is able to *eventually* get the terms of the sequence to be *close enough* to the target, where *close enough* is determined by the chooser. Thus, in order to guarantee a win, no matter what positive distance from the target the chooser decides is close enough (let's call this distance ε), the guesser must be able to find a term in the sequence (call it s_N) such that all of the subsequent terms are within ε of l. The fact that this can be done for every convergent sequence turns out to be the defining property of convergence, which we state formally in Definition 4.1.

Definition 4.1. A sequence $\boldsymbol{s}$ is said to *converge* to a number l provided that for every positive rational number ε, there exists an integer N such that $|s_n - l| < \varepsilon$ for all $n > N$.

As with Informal Definition 2.1, when we say that $\boldsymbol{s}$ *converges* (without specifying a value of l), we mean that there exists some number l such that $\boldsymbol{s}$ converges to l. In this case, we also say that l is the limit of $\boldsymbol{s}$, sometimes written as

$$\lim_{n\to\infty} s_n = l.$$

The next two sections illustrate how Definition 4.1 can be used to prove or disprove the convergence of a sequence of numbers.

Proving Convergence

Consider the sequence $\boldsymbol{s}$ defined by $s_n = 1/n$ for each positive integer n.

Question 4.1. Does $\boldsymbol{s}$ converge? If so, to which number?

Question 4.2. Let l be the limit of $\boldsymbol{s}$ that you found in Question 4.1, and let $\varepsilon = 1/2$. Find an integer N such that $|s_n - l| < \varepsilon$ for all $n > N$, or explain why no such N exists.

Question 4.3. Repeat Question 4.2 for each of the following values of ε:

(a) $\varepsilon = 1/7$

(b) $\varepsilon = .05410728392$

(c) $\varepsilon = 2^{-k} - 1/k$, where k is any integer.

Question 4.4. Generalize your work from Questions 4.2 and 4.3 by writing down a formula for N in terms of ε. In other words, given an arbitrary $\varepsilon > 0$, find a formula for a corresponding N_ε such that $|s_n - l| < \varepsilon$ for all $n > N_\varepsilon$.

Question 4.5. What does your work in Question 4.4 allow you to conclude about the convergence of $\boldsymbol{s}$? Relate your answer to Definition 4.1.

Disproving Convergence

Now consider the sequence $\boldsymbol{t}$ defined by $t_n = \frac{1}{4}(-1)^n$ for each positive integer n.

Question 4.6. Does $\boldsymbol{t}$ converge? If so, to which number?

Question 4.7. Consider the following "proof" that $\boldsymbol{t}$ converges to 0:

> Let $\varepsilon = 1/2$, and let $N = 0$. Since $t_n = -1/4$ for all odd n and $t_n = 1/4$ for all even n, it follows that $|t_n - 0| = 1/4 < \varepsilon$ for all $n > N$. Thus, t_n converges to 0.

Is this proof correct? Why or why not?

Question 4.8. Let $\varepsilon = 1/8$. Is there an integer N such that $|t_n - 0| < \varepsilon$ for all $n > N$? Why or why not?

Question 4.9. Let l be any number, and let $\varepsilon = 1/8$. Show that for every integer N, there exists an integer $n > N$ such that $|t_n - l| \geq \varepsilon$. What does this fact allow you to conclude about the convergence of $\boldsymbol{t}$, and why?

Question 4.10. Suppose that we had defined $\boldsymbol{t}$ so that $t_n = \frac{1}{n}(-1)^n$. Would $\boldsymbol{t}$ have converged in this case? Use Definition 4.1 to thoroughly justify your answer.

Exercises

(1) The definition of convergence of a sequence $\boldsymbol{s}$ is sometimes written in symbolic form as follows:[1]

$$(\exists l)(\forall \varepsilon \in \mathbb{Q}^+)(\exists N \in \mathbb{Z})(\forall n > N)(|s_n - l| < \varepsilon)$$

  (a) Use this symbolic form to write a negation of the definition of convergence. In other words, state, both symbolically and in words, what it means for a sequence $\boldsymbol{s}$ not to converge.

  (b) Describe how your negation from part (a) suggests a strategy for proving that a sequence does not converge. Be specific and precise.

(2) Reconsider the sequence from Questions 2.1 through 2.3 in Activity 2. Use Definition 4.1 to prove or disprove that the sequence converges.

[1] Many texts use $(\forall \varepsilon > 0)$ instead of $(\forall \varepsilon \in \mathbb{Q}^+)$. We prefer the latter, as it more explicitly indicates our universe of discourse, and makes the negation of the definition of convergence more apparent.

(3) Reconsider each of the sequences from Question 2.4 in Activity 2, using Definition 4.1 to prove or disprove the convergence of each sequence.

(4) Revisit Exercise 1 from Activity 2, this time using Definition 4.1 to formally prove your answer.

(5) Revisit Exercise 4 from Activity 2, this time using Definition 4.1 to formally prove your answer.

(6) Revisit Exercise 8 from Activity 2, this time using Definition 4.1 to formally prove your answer.

(7) Let $\boldsymbol{s}$ be a sequence that converges to l. Is the following statement always, sometimes, or never true?

> There exists an integer N such that for every rational number $\varepsilon > 0$, $|s_n - l| < \varepsilon$ for all $n > N$.

Give a proof or a pair of examples (whichever is most appropriate) to justify your answer.

(8) Find (and prove) a necessary and sufficient condition for a sequence to satisfy the property given in Exercise 7.

(9) Suppose that a sequence $\boldsymbol{s}$ does not converge. Is the following property (stated symbolically) necessarily satisfied by $\boldsymbol{s}$?

$$(\exists \varepsilon > 0)(\forall l)(\forall N \in \mathbb{Z})(\exists n > N)(|s_n - l| \geq \varepsilon)$$

Give a proof or counterexample to justify your answer.

(10) Using the same format as Definition 4.1, define in a precise way what it should mean for a sequence $\boldsymbol{s}$ to have an infinite limit. In other words, define

$$\lim_{n\to\infty} s_n = \infty,$$

and give an example (with proof) of a sequence that satisfies your definition.

(11) Prove or disprove: If both $\boldsymbol{s}$ and $\boldsymbol{t}$ converge, then the sequence $\boldsymbol{z}$ defined by

$$z_n = s_n + t_n$$

must also converge.

(12) Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be sequences, and let $\boldsymbol{z}$ be the sequence defined by

$$z_n = s_n + t_n.$$

Prove or disprove each of the following statements:

(a) If $\boldsymbol{z}$ converges, then both $\boldsymbol{s}$ and $\boldsymbol{t}$ must converge.

(b) If $\boldsymbol{z}$ converges, then at least one of $\boldsymbol{s}$ and $\boldsymbol{t}$ must converge.

(c) If $\boldsymbol{z}$ converges and $\boldsymbol{s}$ does not converge, then $\boldsymbol{t}$ must not converge.

(13) Prove or disprove: If a sequence converges, then its limit is unique.

(14) Prove the aptly named *Squeeze Theorem* (or, alternately, the *Sandwich Theorem*):

Theorem 4.1 (Squeeze Theorem)**.** *Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be sequences, and suppose that both $\boldsymbol{a}$ and $\boldsymbol{c}$ converge to the same number, l. Suppose also that there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n > N$. Then $\boldsymbol{b}$ converges to l also.*

(15) Determine the value of each of the following limits, using the Squeeze Theorem to prove each of your answers.

(a) $\displaystyle\lim_{n\to\infty} \frac{\sin(n)}{n}$

(b) $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n}$

(c) $\displaystyle\lim_{n\to\infty} \frac{(-3)^n}{n!}$

Activity 5

Cauchy Sequences and Convergence

Focus Questions

- What is a Cauchy sequence?
- What is Cauchy completeness, and what does the Cauchy Completeness Theorem say about the relationship between Cauchy sequences and convergence?

Introduction

In previous activities, we have discussed the difference between sequences that *accumulate* and sequences that *converge*. In this activity, we will define accumulation in a more precise manner. Doing so will allow us to more carefully investigate the relationship between accumulation and convergence. It will also give us the tools we need in order to formally define and study the real numbers.

Cauchy Sequences

In Activity 2, we considered sequences whose elements can eventually be made as close to each other as we want them to be. We called such sequences *accumulating* sequences, a term that seems to intuitively describe the behavior of sequences whose elements "bunch up" by getting closer and closer to each other.

Historically, however, the notion of accumulation is often attributed to Augustin Cauchy, a 19th century French mathematician who is known for his contributions to many areas of mathematics, including real and complex analysis. Thus, from this point forward, we will follow tradition and refer to accumulating sequences as *Cauchy sequences*. Our formal definition of a Cauchy sequence, which is stated below, is similar in style to our definition of convergence from Activity 4.

Augustin Louis Cauchy (1789 - 1857)

Definition 5.1. A sequence $\boldsymbol{s}$ is said to be a *Cauchy sequence* provided that for every positive rational number ε, there exists an integer N such that $|s_n - s_m| < \varepsilon$ for all $m, n > N$.

Question 5.1. Clearly explain how Definition 5.1 captures the same intuitive idea as our earlier, informal definition of an accumulating sequence.

Question 5.2. Critique the following "proof" that the sequence $\boldsymbol{s}$ defined by $s_n = \sum_{i=1}^{n} \frac{1}{i}$ is Cauchy:

> Let $\varepsilon > 0$ be given, and choose any positive integer N such that $1/N < \varepsilon$. Then for all $n > N$,
>
> $$|s_{n+1} - s_n| = \left| \sum_{i=1}^{n+1} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i} \right| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon.$$
>
> Thus, the elements of $\boldsymbol{s}$ are getting closer to each other, and so $\boldsymbol{s}$ is a Cauchy sequence.

The Equivalence of Cauchy and Convergent

Recall that in Activity 1, we saw an example of a sequence of rational numbers that seemed to accumulate, but that did not converge to a rational number. In other words, we found a Cauchy sequence of rational numbers that did not converge to a rational number. Because of this discovery, we can say that the rational numbers are not *complete*, defined formally as follows:

Definition 5.2. A set S of numbers is said to be *complete* (or *Cauchy complete*) if every Cauchy sequence of elements of S converges to an element of S.

Even after seeing the example from Activity 1, the very existence of number systems that are not complete may seem somewhat counterintuitive. After all, the notions of accumulation and convergence seem very closely related, and it can be hard to visualize how a sequence of numbers can get closer and closer to each other, but not approach some definite value.

The resolution of this apparent paradox ultimately rests on our definition of *number*. Recall that in Activity 1, we restricted ourselves to considering only the rational numbers. In fact, since then, we've made no explicit reference to any numbers other than the rationals. Admittedly, our work would have been much easier if we had admitted the existence of numbers like $\sqrt{2}$. But what is $\sqrt{2}$? This is a surprisingly complex question that we will return to several times throughout the next few activities, but for now we will give a simple answer: $\sqrt{2}$ is a *real* number, and in particular, an *irrational* real number. Intuitively, it is a number that plugs a particular hole that seems to be present in the set of rational numbers. And, as it turns out, when we put all of the real numbers together, we obtain a set that is large enough to plug all such holes. This admittedly imprecise observation translates naturally to a very important theorem about the completeness of the real numbers:

Theorem 5.1 (Cauchy Completeness Theorem)**.** *Let s be a sequence of real numbers. Then s converges to a real number if and only if s is a Cauchy sequence.*

As is the case with most biconditional ("if and only if") statements, our proof of Theorem 5.1 will have two parts – one to establish the "if" direction of the theorem and one to establish the "only if" direction. Unfortunately, neither of these directions can be completed without a more precise definition of the real numbers. (As you might have guessed, "numbers that plug holes in the rationals" is not particularly useful when it comes to writing proofs.) We can, however, prove a special case of the forward implication by once again restricting our attention to the rationals. The result we will prove is the following:

Proposition 5.2. *If s is a sequence of rational numbers that converges to a rational number l, then s is a Cauchy sequence.*

To prove Proposition 5.2, we will first establish the following lemma, which will be of use to us both here and in subsequent activities.

Lemma 5.3 (The Triangle Inequality for $\mathbb{Q}$)**.** *For all rational numbers a and b,*

$$|a + b| \leq |a| + |b|.$$

Question 5.3. Draw a picture to illustrate how the Triangle Inequality gets its name.

Question 5.4. Prove the Triangle Inequality. (Hint: Begin with the fact that $-|x| \leq x \leq |x|$ for every rational number x, including a and b.)

Now that we have established the Triangle Inequality, let's begin our proof of Proposition 5.2. Recall that we are trying to show that if a sequence $\boldsymbol{s}$ of rational numbers converges to a rational number l, then $\boldsymbol{s}$ must be a Cauchy sequence.

Question 5.5. Assume that the sequence $\boldsymbol{s}$ does converge to a rational number l. Is the following statement true or false? Why or why not?

> For every rational number $\varepsilon > 0$, there exists an integer N such that $|s_n - l| < \varepsilon/2$ for all $n > N$.

Question 5.6. Suppose you know that for some integers m and n, $|s_n - l| < \varepsilon/2$ and $|s_m - l| < \varepsilon/2$. What can you then conclude about $|s_n - s_m|$, and why?

Question 5.7. Use your answers to Questions 5.5 and 5.6 to prove that if $\boldsymbol{s}$ converges to a rational number l, then $\boldsymbol{s}$ must be a Cauchy sequence.

Question 5.8. Looking back at your answers to Questions 5.4 through 5.7, identify all of the properties of the rational numbers that you used in your proof of Proposition 5.2. Be thorough and precise, including even those properties that seem obvious to you or that you have taken for granted in the past.

The list of properties you made in Question 5.8 will be very important when the time comes to prove the Cauchy Completeness Theorem in its full generality. In fact, if the properties you used in your proof of Proposition 5.2 can be generalized to the real numbers, then there will be nothing left to prove for the forward implication of the theorem. The reverse implication may put up more of a fight, but the reward – proving that every Cauchy sequence of real numbers converges to a real number – is well worth the effort. We'll tackle this part of the proof in Activity 7, after we have formally defined the real numbers and established some of their most important properties.

Exercises

(1) Consider the following proposition suggested by an undergraduate analysis student:

> Let $\boldsymbol{a}$ be a sequence of rational numbers and let $\boldsymbol{b}$ be the sequence defined by
> $$b_n = a_{n+1} - a_n$$

for each positive integer n. If $\boldsymbol{b}$ converges to zero, then $\boldsymbol{a}$ is a Cauchy sequence.

(a) Critique the following "proof" of this proposition:

> Let $\varepsilon > 0$ be given. Then since $\boldsymbol{b}$ converges to zero, there exists an integer N such that $|b_n - 0| < \varepsilon$ for all $n > N$. Substituting, we then obtain $|a_{n+1} - a_n| < \varepsilon$ for all $n > N$. Now let $m = n + 1$. Then $m, n > N$, and so it follows that
>
> $$|a_n - a_m| = |a_m - a_n| = |a_{n+1} - a_n| < \varepsilon.$$
>
> By definition, however, this means that $\boldsymbol{a}$ is Cauchy.

(b) Is the proposition true? Give a proof or counterexample to justify your answer.

(2) Prove that every Cauchy sequence of rational numbers is bounded. That is, prove that if $\boldsymbol{s}$ is a Cauchy sequence of rational numbers, then there exists a rational number M such that $|s_n| < M$ for all n. (Hint: Reason by contradiction.)

(3) Suppose a sequence $\boldsymbol{s}$ satisfies the property that

$$|s_{n+1} - s_n| < \frac{1}{2^n}$$

for each positive integer n. Is $\boldsymbol{s}$ necessarily a Cauchy sequence? Give a proof or counterexample to justify your answer.

(4) A sequence $\boldsymbol{s}$ is said to be *contractive* provided that there exists a rational number $c \in (0, 1)$ such that

$$|s_{n+2} - s_{n+1}| \le c|s_{n+1} - s_n|$$

for all s_n.

(a) Prove or disprove: Every contractive sequence is Cauchy.

(b) Prove or disprove: Every Cauchy sequence is contractive.

(c) A sequence $\boldsymbol{s}$ is said to be *eventually contractive* provided that there exists a rational number $c \in (0, 1)$ and an integer N such that

$$|s_{n+2} - s_{n+1}| \le c|s_{n+1} - s_n|$$

for all $n > N$. Would either of your answers to parts (a) and (b) have been different if the phrase *eventually contractive* had been used? If so, how? Prove your answers.

(5) Let $\boldsymbol{s}$ be the sequence of partial sums defined by

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}$$

for all $n \geq 0$. (Recall that 0! = 1.) Is $\boldsymbol{s}$ Cauchy? Prove your answer.

(6) Consider each of the sequences defined in part (3) of the *Menu of Sequences* in Appendix A. Which are Cauchy? Prove your answers.

(7) Consider the sequence defined in part (7) of the *Menu of Sequences* in Appendix A. Is this sequence Cauchy? Prove your answer.

(8) Let $\boldsymbol{s}$ be a Cauchy sequence whose elements are all nonzero. Are the following sequences always, sometimes, or never Cauchy sequences? Give a proof or a pair of examples (whichever is most appropriate) to justify your answer.

(a) The sequence $\boldsymbol{t}$ defined by $t_n = s_n^2$.

(b) The sequence $\boldsymbol{w}$ defined by $w_n = 1/s_n$.

(c) The sequence $\boldsymbol{z}$ defined by $z_n = \dfrac{s_{n+1}}{s_n}$.

Activity 6

What is a Real Number?

Introduction

In Activity 5, we introduced the notion of Cauchy completeness, and we used our knowledge the rational numbers to prove a special case of one direction of the Cauchy Completeness Theorem. In order to generalize our work and complete the other direction of the proof, we are going to need to develop a more precise definition of the real numbers.

In this activity, we will consider several definitions of the real numbers that you may have encountered in the past. By discussing the strengths and weaknesses of each of these definitions, we will gain insights into the types of issues that must be addressed by any formal definition of the real numbers.

Informal Definitions of "Real Number"

Consider each of the following informal definitions:

Informal Definition 6.1. A real number is a number that has a finite or infinite decimal expansion.

Informal Definition 6.2. A real number is a point on the number line.

Informal Definition 6.3. A real number is a number that is either rational or the limit of a sequence of rational numbers.

Informal Definition 6.4. A real number is either a rational number or an irrational number, where an irrational number is defined to be one that has an infinite, non-repeating decimal expansion.

Question 6.1. In Activity 5, we said that $\sqrt{2}$ is a real number. Explain how each of Informal Definitions 6.1 – 6.4 could be used to justify this claim. (Hint: You may need to first write down a precise definition of $\sqrt{2}$.)

Question 6.2. For which of Informal Definitions 6.1 – 6.4 was it easiest to argue that $\sqrt{2}$ is a real number? For which was it hardest? For which do you think that your justification was most convincing?

Question 6.3. Discuss in general the strengths and weaknesses of each of Informal Definitions 6.1 – 6.4. Are there situations in which one definition would be easier or more convenient to use than the others? Explain your answers in detail.

Question 6.4. Why do you think that we have referred to the definitions in this activity as "informal" definitions? What makes these definitions different from the "formal" definitions we have adopted in previous activities (for instance, those in Activities 4 and 5)?

Question 6.5. Come up with a definition of the real numbers that is different from those in this activity. Discuss the strengths and weaknesses of your definition, and compare it to Informal Definitions 6.1 – 6.4.

Activity 7

Defining the Real Numbers

Focus Questions

- What is the formal definition of a real number?
- What does it mean for two real numbers to be equal?
- What does it mean for a real number to be positive, or to be negative?
- How can one define addition, subtraction, multiplication, and division of real numbers? What algebraic properties do these operations satisfy?
- What does it mean for one real number to be less than (or greater than) another?
- How can the formal definition of the real numbers be used to prove the Cauchy Completeness Theorem?

Introduction

In Activity 6, we considered several possible definitions of the real numbers. For instance, we said that we could define a real number to be any number that has either a finite or infinite decimal expansion. Or, we could define a real number to be a number that is either rational or the limit of a sequence of rational numbers. In this activity, we'll see that each of these definitions lacks the precision that is necessary for any kind of rigorous analysis involving real numbers. To solve this problem, we will use Cauchy sequences to formally define what a real number is. We will then use our new, formal definition to prove many of the familiar properties of the real numbers that we have taken for granted in the past.

Cauchy Sequences and the Reals

In our very first activity, we saw that Newton's method could generate a sequence of rational numbers whose elements got closer and closer to each other (in other words, an accumulating, or Cauchy, sequence) but that did not converge to another rational number. From our previous experiences, we thought that there should be some number that this sequence converged to. In fact, we thought that the limit of the sequence should be exactly the number that we commonly call $\sqrt{2}$. But what is $\sqrt{2}$? As we noted in Activity 5, this is actually a fairly difficult question to answer, but let's consider a few possibilities:

- We could say that $\sqrt{2} = 1.414213562\cdots$, but what exactly does "$\cdots$" signify? These three little dots would make more sense if the first 9 digits after the decimal point just repeated themselves over and over again, but we've learned in the past that the number we call $\sqrt{2}$ is an irrational number, which implies that its decimal expansion is infinite and non-repeating.

- We could say that $\sqrt{2}$ is the positive solution to the equation $x^2 - 2 = 0$, but how do we know that such a solution exists? If it does, it must live in some set other than the rational numbers, which raises another question: how can we square some unknown quantity, x, that lives in a set we haven't defined yet? While we're at it, what would *positive* mean within the context of this undefined set of numbers?

- Finally, we could say that $\sqrt{2}$ is the limit of the sequence obtained by Newton's method in Activity 1. But then again, how do we know that this limit actually exists? And how can we even talk about the limit of a sequence of numbers that we know cannot converge to a rational number, especially when we haven't yet defined any numbers outside of the rational numbers?

Each of these seemingly intuitive definitions of $\sqrt{2}$ presents difficulties that cannot be resolved unless we adopt a more formal definition of the real numbers. And so that is what we will do here. Be forewarned, however, that the definition we will adopt is not by any means obvious, and it might not seem at all like the picture you have in your mind of what a real number is. Our new definition is, however, equivalent in many ways to the more intuitive definitions that you may have studied in the past. Furthermore, its formality will place our study of the real numbers on a mathematically rigorous foundation, and will even allow us to give good, mathematically precise definitions of familiar numbers like $\sqrt{2}$.

So now, without further ado, here is our formal definition of a real number:

Definition 7.1. A *real number* is a Cauchy sequence of rational numbers.

There are several aspects of this definition that we will need to explore in more detail, but let's begin with the most basic of these: how does this definition of a real number mesh with the other, more intuitive notions that we have considered?

Question 7.1. Each of the following sequences are Cauchy sequences of rational numbers and are thus real numbers according to Definition 7.1. For each sequence, state the common numerical name (for instance, $\sqrt{5}$ or $1/e$ or 15) of the real number defined by the sequence.[1]

(a) $1, 1, 1, \ldots$

(b) 0.1758365, 6.159832, -5.99999, $1, 1, 1, \ldots$

(c) 0, 0.5, 0.57, 0.572, 0.5725, 0.57257, 0.572572, $\ldots$

(d) 3, 3.14, 3.141, 3.1415, 3.14159, $\ldots$

(e) $s_n = \dfrac{1-2n}{3n+4}$

(f) $y_n = \displaystyle\sum_{k=1}^{n} (-1)^k 10^{-k}$

(g) $x_0 = 7/4;\quad x_{n+1} = x_n - \frac{1}{2}(x_n - \frac{3}{x_n})$

(h) $t_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$

Question 7.2. For each of the following real numbers, find a Cauchy sequence of rational numbers that defines the number. Specify each sequence with either a closed formula or a recurrence relation, as in parts (e)–(h) of Question 7.1.

(a) 0

(b) $-\sqrt{5}$ (Hint: Use Newton's Method)

(c) $\dfrac{\pi}{4}$ (Hint: Use the Taylor series for $f(x) = \arctan(x)$ centered at $x = 0$.)

[1] From this point forward, we will sometimes omit quantifiers when defining sequences. Unless specified otherwise, you should assume that the given sequences are defined for each positive integer n.

Now that we have defined what a real number is, there are several natural questions that we will need to answer, such as:

- When are two real numbers equal?
- What does it mean for a real number to be positive or negative?
- How can we add, subtract, multiply, and divide real numbers?
- What does it mean for one real number to be larger or smaller than another?

In the next few sections, we'll consider each of these questions in detail.

Equality of Real Numbers

Question 7.3. Divide the following rational numbers into groups of numbers that are equal to each other:

$$\tfrac{5}{3} \qquad 7.\overline{9} \qquad \tfrac{40}{5} \qquad 1\tfrac{4}{6} \qquad 8.0 \qquad 1.\overline{6}$$

As we were reminded in Question 7.3, any given rational number can be expressed in a variety of ways. Though these various representations are technically different, we consider them to be equivalent because they all correspond to the exact same numerical quantity. In the same way, we will consider different sequence representations of real numbers to be equivalent if they correspond to the same numerical quantity. We'll define this equivalence more precisely in just a moment, but let's first explore the intuition behind when two real numbers should be considered the same.

Question 7.4. Divide the following real numbers into groups of numbers that are equal to each other. Clearly explain the reasoning behind your groupings.

- $1, \frac{3}{2}, 1, \frac{4}{3}, 1, \frac{5}{4}, 1, \ldots$
- $s_0 = -2.9; \quad s_{n+1} = s_n - \dfrac{s_n^2 - 6s_n + 4}{2s_n - 6}$
- $s_0 = 2.9; \quad s_{n+1} = s_n - \dfrac{s_n^2 - 6s_n + 4}{2s_n - 6}$
- $s_0 = 3.1; \quad s_{n+1} = s_n - \dfrac{s_n^2 - 6s_n + 4}{2s_n - 6}$

- $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$

- $s_n = \displaystyle\sum_{k=1}^{n} (-1)^k \frac{1}{k}$

- $s_n = \displaystyle\frac{8}{3} \left(\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \right)^2$

- $s_n = 0.4 + \displaystyle\frac{(-1)^n + 3n}{5n}$

In Question 7.4, you probably said that two real numbers x and y should be considered equal if the elements of the Cauchy sequences that define x and y seem to be approaching the same limit. One way to make this definition more precise is the following:

Definition 7.2. Let x and y be the real numbers defined by the Cauchy sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Then x and y are said to be *equal* if the sequence $\boldsymbol{d}$ defined by $d_n = x_n - y_n$ converges to zero.[2] Stated symbolically,

$$x = y \;\longleftrightarrow\; \lim_{n \to \infty} (x_n - y_n) = 0.$$

Question 7.5. Use Definition 7.2 to prove that the real numbers defined by the sequences in the first and last bullet points of Question 7.4 are equal.

[2]As we have seen, it is possible for two different Cauchy sequences to define the same real number. Thus, when we say that two real numbers x and y are *equal*, we mean that x and y are equal when viewed *as real numbers*, even though the Cauchy sequences that define them may not be equal *as sequences*. The notion of equality, like many other concepts in mathematics, is dependent on the lens through which the mathematical objects in question are viewed. As an example, recall that the rational numbers can be defined as the set of all ordered pairs of integers, with the restriction that the second coordinate is nonzeros. Under this definition, the ordered pairs $(1, 2)$ and $(3, 6)$ would be considered equal *as rational numbers* (since $1/2 = 3/6$), but would not be considered equal when viewed simply as ordered pairs of integers (since $1 \neq 3$ and $2 \neq 6$). We could avoid some of this confusion by defining the rational numbers to be *equivalence classes* of ordered pairs of integers, with equivalence defined in a natural way. Similarly, we could define the real numbers to be equivalence classes of Cauchy sequences of rational numbers; in fact, several other analysis texts do exactly this. We believe, however, that the level of rigor gained from this more formal approach is not sufficient to justify the additional layer of complexity (from both a conceptual and a notational standpoint) required by it. For this reason, we have chosen to avoid the language of equivalence classes and instead discuss equality in more intuitive terms.

Positive and Negative Real Numbers

Question 7.6. For each of the following sequences, decide whether the real number defined by the sequence should be considered positive, negative, or zero. Clearly explain the reasoning behind your decision for each part.

(a) $s_n = \dfrac{1}{n}$

(b) $t_n = \dfrac{(-1)^n}{n}$

(c) $x_n = \dfrac{10-n}{2n-5}$

(d) $y_n = \dfrac{1}{5} - \dfrac{2}{3^n}$

Question 7.6 brings to light several nuances of our definition of a real number, each of which must be taken into consideration before we precisely define what it means for a real number to be positive or negative. Perhaps the most obvious definition of a positive real number would be one for which the elements of the defining Cauchy sequence are all positive. But we have already observed two difficulties with this definition. First, it is possible for the elements of a sequence to start out negative and eventually end up positive. Thus, any definition we adopt will somehow need to allow for sequences that are *eventually* positive or negative. Also, as we've seen, it is possible for sequences whose elements are all positive (or all negative) to still converge to zero. So, for a sequence to be considered positive, what we really need is for the terms of the sequence to be *eventually* positive and *eventually* separated from zero by some nonzero distance. Definition 7.3 incorporates both of these necessary features.

Definition 7.3. Let x be the real number defined by the sequence $\boldsymbol{x}$. Then:

- x is said to be *positive* if there exists a rational number $\alpha > 0$ and an integer N such that $x_n > \alpha$ for all $n > N$.
- x is said to be *negative* if there exists a rational number $\alpha < 0$ and an integer N such that $x_n < \alpha$ for all $n > N$.

Question 7.7. Use Definition 7.3 to prove your answers to Question 7.6.

You may have noticed that we left out what it means for a number to be zero in Definition 7.3. We have done so only because Definition 7.2 already covers this case. (Do you see why?) What does remain to be shown is that every real number must be either positive, negative, or zero (and never more than one of these). This seemingly obvious fact is not an immediate consequence of our definition of positive and negative, and Exercises 2 and 3 at the end of this activity suggest one way to approach its proof.

Operations on Real Numbers

Now that we've defined what a real number is, we need to learn how to perform operations, such as addition and multiplication, on real numbers. Because of the way we have defined the real numbers, all of this boils down to defining how to add, subtract, multiply, and divide Cauchy sequences. In the next few questions, we'll define these operations in the most natural and obvious way possible. We'll then investigate some of the conditions that must hold in order for our operations to work the way we want them to.

Question 7.8. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be the sequences defined by

$$s_n = 2 + \left(-\frac{1}{2}\right)^n \quad \text{and} \quad t_n = \frac{1}{n},$$

respectively. Defining addition, subtraction, multiplication, and division of sequences in the way that seems most natural to you, find a formula for the elements of the sequences $\boldsymbol{s}+\boldsymbol{t}$, $\boldsymbol{s}-\boldsymbol{t}$, $\boldsymbol{s}\cdot\boldsymbol{t}$, and $\boldsymbol{s}\div\boldsymbol{t}$. Will all of these resulting sequences define real numbers? Why or why not?

Question 7.9. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be Cauchy sequences of rational numbers. Define $\boldsymbol{s}+\boldsymbol{t}$, $\boldsymbol{s}-\boldsymbol{t}$, $\boldsymbol{s}\cdot\boldsymbol{t}$, and $\boldsymbol{s}\div\boldsymbol{t}$ in the most natural way you can think of. Under these definitions, are there any conditions on $\boldsymbol{s}$ and $\boldsymbol{t}$ (other than both $\boldsymbol{s}$ and $\boldsymbol{t}$ being Cauchy) that must hold in order for these operations to make sense? Are there any conditions that must hold in order for these operations to be guaranteed to always result in another Cauchy sequence?

By defining how to add, subtract, multiply, and divide Cauchy sequences of rational numbers in Question 7.9, we have actually defined how to add, subtract, multiply, and divide real numbers. Under the most natural way of defining these operations (which you likely came up with in Question 7.9), the real numbers turn out to be a *field*, which means that they satisfy all of the following familiar properties:

- **Closure under addition and multiplication:** For all $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$ and $x \cdot y \in \mathbb{R}$.

- **Associativity of addition:** For all x, y, $z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.
- **Commutativity of addition:** For all x, $y \in \mathbb{R}$, $x + y = y + x$.
- **Existence of an additive identity:** There exists a real number e_+ such that for all $x \in \mathbb{R}$, $x + e_+ = e_+ + x = x$.
- **Existence of additive inverses:** For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y = y + x = e_+$.
- **Associativity of multiplication:** For all x, y, $z \in \mathbb{R}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- **Commutativity of multiplication:** For all x, $y \in R$, $x \cdot y = y \cdot x$.
- **Existence of a multiplicative identity:** There exists a real number $e_\times$ such that for all $x \in \mathbb{R}$, $x \cdot e_\times = e_\times \cdot x = x$.
- **Existence of multiplicative inverses:** For all nonzero $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x \cdot y = y \cdot x = e_\times$.
- **Distribution of multiplication over addition:** For all x, y, $z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

You will prove many of these properties in the exercises at the end of the activity. For now, however, we'll proceed with our investigations of the real numbers by using the operations we just defined, together with our notions of positive and negative, to define what it means for one real number to be larger or smaller than another.

Ordering the Real Numbers

Fill in the blanks to complete the following definition in a way that is consistent with your previous understanding of the "less than" and "greater than" relations.

Definition 7.4. Let x and y be real numbers. Then:

- x is said to be *less than* y, denoted $x < y$, provided that $x - y$ is ___________.
- x is said to be *greater than* y, denoted $x > y$, provided that $x - y$ is ___________.

Question 7.10. Use your answer to Question 7.2, part (c), along with the definitions of positive, subtraction, and less than, to prove that $\pi/4 < 1$.

In Activity 5, we stated the Triangle Inequality for rational numbers, but the same result holds for real numbers as well.

Question 7.11. Use Definition 7.4 and the other properties we have developed in this activity to write a rigorous proof of the Triangle Inequality for real numbers. Thoroughly justify each step of your proof using only the properties of the real numbers that we have stated or proved up to this point.[3]

Completeness of the Real Numbers

In this section, we will finally arrive at the final destination of our formal study of the definition of the real numbers. Recall that in Activity 5, we showed that every convergent sequence of rational numbers is also a Cauchy sequence. Here, we will show that the converse of this statement is true for all sequences of real numbers. That is, we will show that every Cauchy sequence of real numbers must converge to a real number. We will then discuss how our work in Activity 5 can be generalized in order to finish the proof of the Cauchy Completeness Theorem.

We will begin with the following lemma, which essentially states that every real number can be approximated arbitrarily well by a rational number. In more technical terms, this property means that the rational numbers are *dense* in the reals.

Lemma 7.1. *For every real number x and every rational number $\varepsilon > 0$, there exists a rational number q such that $|x - q| < \varepsilon$.*

Question 7.12. Prove Lemma 7.1. (Hint: Argue that if $\boldsymbol{x}$ is a Cauchy sequence of rational numbers, then for every rational $\varepsilon > 0$, there exists an integer k such that $|x_k - x_n| < \varepsilon/2$ for all $n \geq k$. Let $q = x_k$, and argue that $|x - q| < \varepsilon$.)

Now that we have established Lemma 7.1, we are ready to move on to our proof that every Cauchy sequence of real numbers converges to a real number. Question 7.13 suggests one possible strategy for this proof.

Question 7.13. Let $\boldsymbol{x}$ be a Cauchy sequence of real numbers.

(a) For every positive integer n, choose a rational number q_n such that $|x_n - q_n| < 1/n$. Use Lemma 7.1 to explain why such a number exists.

[3] Note that absolute value is defined for real numbers in the exact same way that it is defined for rational numbers; that is, $|x| = x$ if x is positive or zero, and $|x| = -x$ if x is negative.

(b) Show that the sequence $\boldsymbol{q}$ defined by the q_n from part (a) is a Cauchy sequence of rational numbers. Deduce then that $\boldsymbol{q}$ defines a real number, say L. (Hint: Note that

$$|q_m - q_n| = |q_m - x_m + x_m - x_n + x_n - q_n|.$$

Use this identity, along with part (a), the Triangle Inequality, and the fact that $\boldsymbol{x}$ is a Cauchy sequence.)

(c) Show that the sequence $\boldsymbol{x}$ converges to L. (Hint: The fact that L is defined by the sequence $\boldsymbol{q}$ implies that $|q_n - L| \to 0$ as $n \to \infty$. Use this fact, along with part (a), the identity

$$|x_n - L| = |x_n - q_n + q_n - L|,$$

and the Triangle Inequality.)

Question 7.13 wraps up the proof of the reverse implication of the Cauchy Completeness Theorem, which leaves only the easier forward implication. Recall that we proved a special case of this part of the theorem in Activity 5. When we did so, we also made a list of properties of the rational numbers that we used in our proof.

Question 7.14. Look back again at the proof you wrote in Questions 5.4 through 5.7, and the list of properties you made in Question 5.8. Which of these properties also hold in the real numbers? Prove as many properties as you can, and use your work to explain why the proof you wrote in Activity 5 can be generalized to include sequences of real numbers.

Revisiting $\sqrt{2}$

Recall that, at the beginning of this activity, we considered three possible ways of defining $\sqrt{2}$, each of which revealed inadequacies in our informal definitions of the real numbers. To conclude our investigations here, let's revisit the problem of defining $\sqrt{2}$, this time using our formal, Cauchy sequence definition of the real numbers. Doing so will demonstrate both the benefit and the necessity of this more formal approach.

Question 7.15. Use your work from this activity to prove that there is a positive real number x such that $x^2 - 2 = 0$. (Hint: Use the sequence defined by Newton's method in Activity 1. You've already done most of the work there!)

One Final Note

In Definitions 4.1 and 5.1, we let ε denote an arbitrary positive rational number. In other texts, however, these definitions are usually stated using real values of ε. As it turns out, Lemma 7.1 implies that the two different formulations are completely equivalent. Now that we have formally defined and studied the real numbers, we will from this point forward use the more traditional definitions of convergence and Cauchy (*i.e.*, those that allow both rational and irrational values of ε). We will also allow ourselves to refer to familiar real numbers such as π and e without actually proving that these numbers are real.

Exercises

Many of the exercises that follow establish properties that we stated, but did not prove, throughout this activity. The proof of the Cauchy Completeness Theorem relies on several of these properties. Thus, to avoid circular reasoning, you should complete the exercises below without appealing to the Cauchy Completeness Theorem. Instead, justify your reasoning using only the definitions and properties stated or proved in this activity.

(1) Prove that the equals relation on the real numbers is an equivalence relation. In other words, prove that:

- = is **reflexive**: For every $x \in \mathbb{R}$, $x = x$.
- = is **symmetric**: For all $x, y \in \mathbb{R}$, if $x = y$, then $y = x$.
- = is **transitive**: For all $x, y, z \in \mathbb{R}$, if $x = y$ and $y = z$, then $x = z$.

(2) Prove that every real number is either positive, negative, or zero. (Hint: Prove that if x is a real number and $x \neq 0$, then x is either positive or negative.)

(3) Prove that a real number cannot be...

(a) ...both positive and negative.

(b) ...both positive and zero.

(c) ...both negative and zero.

(4) Prove that for all $x, y \in \mathbb{R}$, either $x < y$, $x = y$, or $x > y$. (Hint: Use Exercise 2.)

(5) Prove that, for all x, $y \in \mathbb{R}$, it cannot be the case that...

  (a) ...$x < y$ and $x > y$.

  (b) ...$x = y$ and $x > y$.

  (c) ...$x = y$ and $x < y$.

(6) Let $x \in \mathbb{R}$. Prove that x is negative if and only if the additive inverse of x is positive.

(7) Let $x, y \in \mathbb{R}$. Prove that if x and y are both positive, then $x + y$ and $x \cdot y$ are both positive.

(8) Prove that the real numbers are closed under addition. That is, prove that if x and y are real numbers, then $x + y$ is also a real number.

(9) Prove that the real numbers are closed under multiplication. (Hint: You will need to use Exercise 2 from Activity 5.)

(10) Prove that addition of real numbers is well-defined. That is, prove that for all x, y, a, $b \in \mathbb{R}$, if $x = a$ and $y = b$, then $x + y = a + b$.

(11) Prove that multiplication of real numbers is well-defined. That is, prove that for all x, y, a, $b \in \mathbb{R}$, if $x = a$ and $y = b$, then $x \cdot y = a \cdot b$.

(12) Prove that addition of real numbers is associative.

(13) Prove that addition of real numbers is commutative.

(14) Prove that the real numbers contain an additive identity.

(15) Prove that each real number has an additive inverse in the real numbers.

(16) Prove that multiplication of real numbers is associative.

(17) Prove that multiplication of real numbers is commutative.

(18) Prove that the real numbers contain a multiplicative identity.

(19) Prove that a real number x has a multiplicative inverse in the real numbers if and only if $x \neq 0$.

(20) Prove that multiplication distributes over addition in the real numbers.

(21) Prove that for all $x \in \mathbb{R}$, $0x = 0$.

(22) Prove that the less than ($<$) relation on the real numbers is well-defined. That is, prove that for all x, y, a, $b \in \mathbb{R}$, if $x = a$ and $y = b$, then $x < y$ implies $a < b$.

(23) Let $\leq$ be the relation on the real numbers defined in the usual way. (That is, $x \leq y$ if and only if $x < y$ or $x = y$.) Show that $\leq$ is a partial order on $\mathbb{R}$. In other words, show that:

- $\leq$ is **reflexive**: For all $x \in \mathbb{R}$, $x \leq x$.
- $\leq$ is **transitive**: For all x, y, $z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- $\leq$ is **antisymmetric**: For all $x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.

(24) Let $\boldsymbol{x}$ be a sequence of positive real numbers, and suppose that $\boldsymbol{x}$ converges to a real number L. Must $L \geq 0$? Must $L > 0$? Prove your answers.

(25) Let x be the real number defined by the sequence $\boldsymbol{x}$, and suppose that for some rational $\varepsilon > 0$, $-\varepsilon < x_n < \varepsilon$ for all n. Is it necessarily the case that $|x| < \varepsilon$? If so, prove it. Otherwise, find a counterexample **and** prove a related result that is true.

(26) (a) Prove that for any real numbers a and b with $b > a$, there is another real number between a and b.

(b) Use part (a) to deduce that for any real numbers a and b with $b > a$, there are infinitely many real numbers between a and b.

(27) Prove that for any real numbers a and b with $b > a$, there are infinitely many *rational* numbers between a and b.

Activity 8

Boundedness, Monotonicity, and Sub-sequences

Focus Questions

- What does it mean for a sequence to be bounded above and/or bounded below?
- What does it mean for a sequence to be monotone?
- What does it mean for one sequence to be a sub-sequence of another?

Introduction

In Activity 7, we proved that every Cauchy sequence of real numbers must converge to a real number. In doing so, we discovered that one way to prove that a sequence of real numbers is convergent is to prove that it is Cauchy. In this activity, we will explore several other important properties of sequences, each of which can play an important role in proving or disproving the convergence of sequences of real numbers.

Boundedness and Monotonicity

Question 8.1. Use the sequences from the *Menu of Sequences* in Appendix A to answer each of the following questions.

(a) Which of the sequences are **bounded above**? That is, for which of the sequences is there a real number u (called an *upper bound*) such that u is at least as large as every element of the sequence?

(b) Which of the sequences are **bounded below**? That is, for which of the sequences is there a real number l (called a *lower bound*) such that l is at least as small as every element of the sequence?

(c) Which of the sequences are **never increasing**?[1] That is, which sequences $\boldsymbol{x}$ satisfy the condition that $x_{n+1} \leq x_n$ for all n?

(d) Which of the sequences are **never decreasing**? That is, which sequences $\boldsymbol{x}$ satisfy the condition that, $x_{n+1} \geq x_n$ for all n?

(e) Decide whether each of the sequences on the menu converges or does not converge. You do not need to give formal proofs of your answers, but you should give a brief justification for each.

(f) We often say that a sequence is *monotone* if it is either never increasing or never decreasing. Do your answers to parts (a)–(e) suggest any results about the convergence of sequences that are bounded and/or monotone? Make as many conjectures as you can.

Sub-sequences

Let $\boldsymbol{s}$ be the sequence (of real numbers) defined by

$$s_n = \begin{cases} 1/n, & \text{if } n \text{ is odd} \\ 7 + 1/n, & \text{if } n \text{ is even} \end{cases},$$

and consider the sequence $\boldsymbol{t}$ defined by

$$\begin{aligned} t_1 &= 1 \\ t_2 &= 7 + 1/4 \\ t_3 &= 1/7 \\ t_4 &= 7 + 1/10 \\ &\vdots \end{aligned}$$

[1] Note that many texts use the terms *non-increasing* and *non-decreasing* instead of *never increasing* and *never decreasing*. Note also that a constant sequence is both never increasing and never decreasing. A sequence for which $x_{n+1} > x_n$ for all n is said to be *increasing* (or *always increasing*), whereas a sequence for which $x_{n+1} < x_n$ for all n is said to be *decreasing* (or *always decreasing*.)

Question 8.2. Describe in a mathematically precise way the relationship between $\boldsymbol{s}$ and $\boldsymbol{t}$.

When two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ have a relationship like that of sequences $\boldsymbol{s}$ and $\boldsymbol{t}$ from the previous example, we often say that $\boldsymbol{y}$ is a sub-sequence of $\boldsymbol{x}$. In other words:

Informal Definition 8.1. If $\boldsymbol{x}$ is a sequence and $\boldsymbol{y}$ is a sequence that contains only elements from $\boldsymbol{x}$ in the same order as they appear in $\boldsymbol{x}$, then $\boldsymbol{y}$ is said to be a *sub-sequence* of $\boldsymbol{x}$.

Or, capturing the same idea in a slightly more formal manner, we could say the following:

Definition 8.1. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be a sequences of real numbers. Then $\boldsymbol{y}$ is said to be a *sub-sequence* of $\boldsymbol{x}$ provided that there exists a sequence $\boldsymbol{k}$ of integers such that $k_1 < k_2 < k_3 < \cdots$ and $y_n = x_{k_n}$ for all n.

Question 8.3. Consider once again the sequences $\boldsymbol{s}$ and $\boldsymbol{t}$ defined on the previous page.

(a) Is $\boldsymbol{s}$ a convergent sequence? Does $\boldsymbol{s}$ have a sub-sequence or sub-sequences that are convergent?

(b) Is $\boldsymbol{t}$ a convergent sequence? Does $\boldsymbol{t}$ have a sub-sequence or sub-sequences that are convergent?

(c) Let $\boldsymbol{z}$ be any sequence. Does $\boldsymbol{z}$ necessarily have at least one convergent sub-sequence? Give a proof or counterexample to justify your answer.

(d) Let $\boldsymbol{z}$ be a never increasing sequence. Must $\boldsymbol{z}$ be convergent? Must $\boldsymbol{z}$ have a convergent sub-sequence? Give a proof or counterexample to justify each of your answers.

(e) Suppose that in parts (c) and (d), we had also required $\boldsymbol{z}$ to be bounded above and below. How, if at all, would this requirement have changed your answers, and why?

Activity 9

The Bolzano-Weierstrass Theorem

Focus Questions

- What does the Bolzano-Weierstrass Theorem say about bounded sequences?
- What is the supremum of a set of real numbers? What does the Dedekind Completeness Theorem say about the existence of suprema?
- How can the Dedekind Completeness Theorem be used to prove the Bolzano-Weierstrass Theorem?

Introduction

In our last activity, we explored the properties of boundedness and monotonicity for sequences of real numbers. In this activity, we will prove the Bolzano-Weierstrass Theorem, an important and useful result about bounded sequences. Along the way, we will discover several other important theorems about sequences, some of which you may have conjectured yourself during our initial investigations into boundedness, monotonicity, and sub-sequences.

The Main Result and Our Proof Strategy

The main result that we will prove in this activity is the following theorem, named after Bernard Placidus Johann Nepomuk Bolzano, an Austrian mathematician, priest, and philosopher, and Karl Theodor Wilhelm Weierstrass, a German mathematician who has been called the "founder of modern analysis."

Bernard Placidus Johann Nepomuk Bolzano
(1781-1848)

Karl Theodor Wilhelm Weierstrass
(1815-1897)

Theorem 9.1 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence (that is, every sequence that is both bounded above and bounded below) has a convergent sub-sequence.*

We will prove the Bolzano-Weierstrass Theorem through a sequence of intermediate results, many of which are important and significant by themselves. Our general strategy will be to first prove that every bounded, monotone sequence must converge. We will then argue that every bounded sequence contains a subsubsequence that is monotone (and of course bounded), and thus convergent.

Bounded and Monotone Sequences

The first result in our journey toward the Bolzano-Weierstrass Theorem is the following lemma:

Lemma 9.2. *If $\boldsymbol{x}$ is a sequence of real numbers that is both bounded and monotone, then $\boldsymbol{x}$ converges.*

When thinking about Lemma 9.2, it is important to keep in mind that a sequence is *bounded* if and only if it is bounded above **and** bounded below. Also recall that a sequence is monotone if and only if it is either never increasing **or** never decreasing.

Question 9.1. Suppose that $\boldsymbol{x}$ is a never decreasing sequence, and suppose also that $\boldsymbol{x}$ does not converge.

(a) Prove that for some $\varepsilon > 0$, there exists a sub-sequence $x_{k_1}, x_{k_2}, x_{k_3}, \cdots$ of $\boldsymbol{x}$ such that $x_{k_{n+1}} > x_{k_n} + \varepsilon$ for all n. (Hint: Use the contrapositive of the Cauchy Completeness Theorem, being careful to correctly negate the definition of a Cauchy sequence.)

(b) Explain how your proof from part (a) implies that $\boldsymbol{x}$ is not bounded.

(c) Explain how your proof from part (a) could be modified to account for the case that $\boldsymbol{x}$ is never increasing.

(d) Explain how your work in parts (a)–(c) establishes Lemma 9.2.

With Lemma 9.2 in hand, let's now move on to the next step in our proof of the Bolzano-Weierstrass Theorem. To complete the proof, we will first need to consider the notion of the least upper bound of a set of real numbers.

Least Upper Bounds and Dedekind Completeness

Question 9.2. Consider the set S of real numbers defined by

$$S = \{x \in \mathbb{R} : \ln(x) < 1\}.$$

(a) Find at least three different upper bounds for S.

(b) Does S have a *least upper bound*? That is, is there a real number u such that (i) u is an upper bound for S (that is, $u \geq x$ for all $x \in S$); and (ii) if u' is an upper bound for S, then $u' \geq u$?

The notion of the least upper bound, or *supremum*, of a set of real numbers is an important idea that is closely related to our earlier investigations of Cauchy sequences and completeness. We will formally define the supremum of a set as follows:

Definition 9.1. Let S be a set of real numbers. The *supremum* of S, denoted $\sup(S)$, is the smallest real number that is an upper bound for S.

The more detailed version of Definition 9.1 is exactly the one given in part (b) of Question 9.2 above. That is, the supremum of S is a real number u such that (i) u is an upper bound for S (that is, $u \geq x$ for all $x \in S$); and (ii) if u' is an upper bound for S, then $u' \geq u$. Note that the *infimum*, or *greatest lower bound*, of a set of real numbers can be defined in a completely analogous manner.

It's important to note that by defining the supremum of S to be "the smallest" real number that is an upper bound for S, we are implicitly assuming two things:

first, that there is a smallest upper bound, and second, that this smallest upper bound is unique. (Using "the" instead of "a" suggests uniqueness.) We should not take either of these facts for granted. In fact, the reason we are discussing least upper bounds right now is because their existence will allow us to construct the bounded, monotone sub-sequence that we need to complete the proof of the Bolzano-Weierstrass Theorem. With that in mind, our next step will be to prove the following existence theorem, leaving the uniqueness argument as an exercise.

Theorem 9.3 (Dedekind Completeness Theorem)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Question 9.3. Give an example to show that the assumption of boundedness is an essential part of Theorem 9.3.

Theorem 9.3 is named after Julius Wilhelm Richard Dedekind, a German mathematician who is most widely known for his approach to the construction of the real numbers using sets called *Dedekind cuts*. This approach is different, but ultimately equivalent, to our approach using Cauchy sequences. And just as our approach led to the Cauchy Completeness Theorem (which, as you will recall, states that every Cauchy sequence of real numbers converges to a real number), Dedekind's approach leads to a similar notion of completeness, one that turns out to be logically equivalent to Cauchy completeness.

Julius Wilhelm Richard Dedekind
(1831-1916)

For now, we will prove only the direction of this equivalence that is necessary in order to finish our proof of the Bolzano-Weierstrass Theorem. That is, we will use the Cauchy Completeness Theorem (which we have already proved) to prove the Dedekind Completeness Theorem. The corresponding reverse implication is given as an exercise at the end of this activity.

In order to prove the Dedekind Completeness Theorem, we will need one additional lemma:

Lemma 9.4. *Let $\boldsymbol{x}$ be a sequence of real numbers, and suppose that $\boldsymbol{x}$ converges to a real number L. Then every sub-sequence of $\boldsymbol{x}$ must also converge to L.*

Question 9.4. Prove Lemma 9.4. (Hint: Begin by choosing an arbitrary sub-sequence $\boldsymbol{y}$ of $\boldsymbol{x}$. Then use the fact that $\boldsymbol{x}$ converges to L to show that $\boldsymbol{y}$ must also converge to L. This latter step can be completed using either a direct proof or a proof by contradiction.)

To prove the Dedekind Completeness Theorem, we will begin by letting S be any nonempty set of real numbers that is bounded above. We will then construct a Cauchy sequence that converges to a supremum for S in the following manner: let r_1 be an upper bound for S, let s_1 be any element of S, and let $a_1 = \frac{r_1+s_1}{2}$. Define the sequences $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{a}$ recursively as follows:

- If a_n is not an upper bound for S, then choose s_{n+1} to be any element of S that is greater than a_n, and let $r_{n+1} = r_n$.
- If a_n is an upper bound for S, then let $r_{n+1} = a_n$ and $s_{n+1} = s_n$.
- In either of the above cases, let $a_n = \frac{r_n+s_n}{2}$. (In other words, let a_n be the midpoint, or average, of r_n and s_n.)

Question 9.5. Let $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{a}$ be a defined above.

(a) Use Lemma 9.2 to prove that both $\boldsymbol{r}$ and $\boldsymbol{s}$ converge.

(b) Use part (a) to deduce that the sequence $\boldsymbol{a}$ also converges.

(c) Prove that $\boldsymbol{r}$ and $\boldsymbol{s}$ must converge to the same limit. (Hint: Show that $r_{n+1} - s_{n+1} \leq \frac{1}{2}(r_n - s_n)$ for all n.)

(d) Let $u = \lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n$. Prove that u is an upper bound for S.

(e) Prove that if $u' < u$, then u' is not an upper bound for S. (Hint: Use the fact that $\boldsymbol{s}$ converges to u to find an element $x \in S$ such that $u' < x \leq u$.) Deduce that u is a least upper bound for S.

Completing Our Proof

Now that we have established the Dedekind Completeness Theorem, we are finally able to complete our proof of the Bolzano-Weierstrass Theorem. Recall that we are trying to show that every bounded sequence of real numbers has a convergent subsequence. Thus, let $\boldsymbol{x}$ be any bounded sequence, and define the set S as follows:

$$S = \{z \in \mathbb{R} : \text{finitely many elements of } \boldsymbol{x} \text{ are less than } z\}$$

Question 9.6. Let $\boldsymbol{x}$ and S be as defined above.

(a) Argue that S is nonempty and bounded above, and thus S has a least upper bound, say u.

(b) Suppose $u \in S$. Prove that, in this case, there must exist a sub-sequence of $\boldsymbol{x}$ that converges to u. (Hint: Begin by showing that for every $\varepsilon > 0$, there exist infinitely many elements of $\boldsymbol{x}$ between u and $u + \varepsilon$. Then choose successively smaller values of ε.)

(c) Suppose $u \notin S$. Prove that, in this case, there must also exist a sub-sequence of $\boldsymbol{x}$ that converges to u. (Hint: In a manner similar to part (b), begin by showing that for every $\varepsilon > 0$, there exist infinitely many elements of $\boldsymbol{x}$ between $u - \varepsilon$ and u.)

(d) Explain how your work in parts (a)–(c) establishes the Bolzano-Weierstrass Theorem.

Exercises

(1) Prove that the supremum of a set of real numbers is unique.

(2) Use the Dedekind Completeness Theorem to prove that every Cauchy sequence of real numbers converges to a real number. Deduce that Dedekind completeness and Cauchy completeness are equivalent properties of the real numbers.

(3) Prove that the Bolzano-Weierstrass Theorem implies Lemma 9.2.

(4) What does the Bolzano-Weierstrass Theorem allow you to conclude about each of the sequences on the *Menu of Sequences* in Appendix A, and why?

Activity 10

Preserving Convergence

Focus Questions

- What does it mean for a function to preserve convergence?
- What types of functions preserve convergence?

Introduction

Mathematicians often view mathematics as the study of precisely defined objects and the functions that preserve certain properties of these objects. As such, the study of functions is a common theme throughout virtually every area of mathematics. For example, in linear algebra, linear transformations are important because they preserve linear combinations of vectors in vector spaces. In abstract algebra, homomorphisms are important because they preserve the binary operations defined on structures such as groups, rings, and fields. In geometry, isometries are important because they preserve distance, and in graph theory, isomorphisms are important because they preserve adjacency.

In light of these examples, it is natural to ask which types of functions are important to the study of analysis. Since our investigations thus far have been focused on sequences, and since convergence is arguably the most important property related to sequences, we will attempt to classify the types of functions that preserve the convergence of sequences of real numbers.

Preserving Convergence

For the following definitions, let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function.

Definition 10.1. Let $\boldsymbol{x}$ be a sequence of real numbers, all of which are elements of D. Then the *image* of $\boldsymbol{x}$ under f, denoted $f(\boldsymbol{x})$, is the sequence $\boldsymbol{y}$ defined by

$$y_n = f(x_n)$$

for all n.

Definition 10.2. Let $\boldsymbol{x}$ be a sequence of real numbers, all of which are elements of D, and suppose that $\boldsymbol{x}$ converges to some real number a. Then f is said to *preserve the convergence* of $\boldsymbol{x}$ provided that $a \in D$ and $f(\boldsymbol{x})$ converges to $f(a)$.

Definition 10.3. The function f is said to *preserve convergence* at a if $a \in D$ and f preserves the convergence of every sequence of elements of D that converges to a.

Question 10.1. For each of the functions listed below, decide whether the function preserves convergence at 0 and/or at 2. Give a brief explanation or counterexample to justify each of your answers.

(i)

$$g : x \to \begin{cases} \frac{1}{x}, & x \neq 0 \\ 5, & x = 0 \end{cases}$$

(ii)

$$h : x \to \begin{cases} \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(iii)

$$l : x \to \frac{x^2 - 4}{x - 2}, x \neq 2$$

(iv)

$$f : x \to \begin{cases} x + 3, & x \neq 0, x \neq 2 \\ 3, & x = 0 \\ 4, & x = 2. \end{cases}$$

(v)

$$m : x \to \lfloor x \rfloor$$ [2]

(vi)

$$n : x \to \left\lfloor x + \frac{1}{2} \right\rfloor$$

(vii)

$$k : x \to \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(viii)

$$p : x \to \begin{cases} |x|, & x \neq 2 \\ 2, & x = 2 \end{cases}$$

[2]This is the "floor" or "round down" function. It returns an integer if the input is an integer; otherwise it returns the integer immediately below the input. So, for example, $\lfloor 2.35 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, and $\lfloor -2.35 \rfloor = -3$.

Question 10.2. Looking back at your work in Question 10.1, make a conjecture about the types of functions that preserve convergence at a. What familiar property is sufficient to guarantee that a function will preserve convergence?

Activity 11

Continuity and Limits of Functions

Focus Questions

- What does it mean for a function to be continuous at a point?
- What is the precise definition of the limit of a function at a point?
- What is the relationship between continuity and the property of preserving convergence?

Introduction

In Activity 10, we learned about what it means for a function to preserve convergence. What you may have conjectured then is that a function f preserves convergence at a point a if and only if f is continuous at a. In other words, the new notion of preserving convergence is actually equivalent to the more familiar notion of continuity. In this activity, we will explore this connection in more detail by precisely defining continuity and investigating some important properties of continuous functions. Along the way, we will also formally define the limit of a function at a point, and we will see how this formal definition relates to other, more intuitive ideas about limits.

Continuous Functions

You may recall the following definition of continuity from your first-semester calculus class:

Informal Definition 11.1. Let f be a function and let a be any real number. Then f is said to be *continuous at* a if all three of the following conditions are satisfied:

- a is in the domain of f. (In other words, $f(a)$ is defined.)
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$.

The intuitive idea behind this definition is that for a function f to be called continuous at a point a, the behavior of f *near* a should be similar to the behavior of f *at* a. As you can see, we can use limits to make this intuitive idea more precise. However, while we have formally defined what the limit of a sequence of real numbers is, we haven't yet formally defined the limit of a function at a point. To do so, we'll start with a less formal definition, which is similar in format to our informal definition of the limit of a sequence.

Limits of Functions

Informal Definition 11.2. Let f be a function and let a be any real number. We say that the *limit* of $f(x)$ as x approaches a is equal to L, denoted

$$\lim_{x \to a} f(x) = L,$$

provided that $f(x)$ can be made arbitrarily close to L (as close to L as we want) by choosing x sufficiently close (close enough) to, but not equal to, a.

Question 11.1. Let $f(x) = 7x - 4$.

(a) Find a value of L for which $\lim_{x \to 1} f(x) = L$.

(b) Suppose you wanted $f(x)$ to be within a distance of 0.1 from the value of L you found in part (a). How close to 1 would x need to be in order to make this happen?

(c) Suppose you wanted $f(x)$ to be within a distance of 0.01 from the value of L you found in part (a). How close to 1 would x need to be in this case?

(d) Let $\varepsilon > 0$ be any real number. How close to 1 does x need to be in order to guarantee that $f(x)$ will be within a distance of ε from the value of L you found in part (a)?

(e) Use Informal Definition 11.2 to explain why $\lim_{x \to 1} f(x) \neq 3.01$.

Question 11.1 suggests the following formal definition of the limit of a function at a point:

Definition 11.1. Let $f : D \to \mathbb{R}$ be a function, and let a be any real number. We say that the *limit* of $f(x)$ as x approaches a is equal to L, denoted

$$\lim_{x \to a} f(x) = L,$$

provided that for every positive real number ε, there exists a positive real number δ such that for all $x \in D$,

$$0 < |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Note that Definition 11.1 can be written symbolically as follows:[1]

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in D)(0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon)$$

Also note that if there is no real number L for which $\lim_{x \to a} f(x) = L$, then we say that the limit of $f(x)$ at a *does not exist*.

Question 11.2.

(a) Explain how each of the parts of Definition 11.1 corresponds to a part of Informal Definition 11.2.

(b) Use the symbolic form of Definition 11.1 to write down its negation.

(c) Use Definition 11.1 to prove that for any real numbers a, m, and b,

$$\lim_{x \to a} (mx + b) = ma + b.$$

(Hint: Begin by choosing an arbitrary $\varepsilon > 0$. Then find a corresponding $\delta > 0$ (which should depend on ε) so that $|(mx + b) - (ma + b)| < \varepsilon$ whenever $0 < |x - a| < \delta$.)

[1]Many texts use $(\forall \varepsilon > 0)$ and $(\exists \delta > 0)$ in the symbolic definition of the limit. As noted earlier (see page 17), we prefer to more explicitly state our universe of discourse.

(d) Use Definition 11.1 to prove that for any real number a, $\lim_{x\to a} x^2 = a^2$.

(e) Let f be defined as follows:

$$f(x) = \begin{cases} x^3 & \text{if } x \neq \pi \\ 31 & \text{if } x = \pi \end{cases}$$

Does $\lim_{x\to\pi} f(x)$ exist? If so, what is its value? Use Definition 11.1 to prove your answer.

(f) Use Definition 11.1 to prove or disprove that $\lim_{x\to\pi} e^x = \frac{162}{7}$.

Back to Continuity

Now that we understand the formal definition of the limit of a function at a point, we can adopt a similar definition of continuity:

Definition 11.2. Let $f : D \to \mathbb{R}$ be a function, and let a be any real number. Then f is said to be *continuous* at a provided that $a \in D$, and for every positive real number ε, there exists a positive real number δ such that for all $x \in D$,

$$|x - a| < \delta \text{ implies } |f(x) - f(a)| < \varepsilon.$$

If f is continuous at every point in some set $S \subseteq D$, then f is said to be *continuous on* S. If f is continuous at every point in its domain, then f is simply said to be *continuous*.

Question 11.3.

(a) Explain how each of the parts of Definition 11.2 corresponds to a part of Informal Definition 11.1.

(b) Why does the definition of continuity use the hypothesis that $|x - a| < \delta$ instead of $0 < |x - a| < \delta$, as in Definition 11.1?

(c) Look back at parts (c)–(e) of Question 11.2. In light of Definition 11.2, rephrase what you proved in each of these parts using the language of continuity.

Continuity and Preserving Convergence

Let's now conclude our investigations of continuity by proving the equivalence that we suggested at the beginning of this activity, which we can now state formally as the following theorem:

Theorem 11.1. *Let f be a function and let a be any real number. Then f is continuous at a if and only if f preserves convergence at a.*

Question 11.4. Prove Theorem 11.1. (Hints: For the "only if" direction, let $\boldsymbol{x}$ be any sequence that converges to a. Use the ε-δ definition of continuity along with the ε-N definition of convergence to show that $f(\boldsymbol{x})$ must converge to $f(a)$. For the "if" direction, use a proof by contrapositive. That is, assume that f is not continuous at a. Then use the negation of Definition 11.2 to construct a sequence $\boldsymbol{x}$ that converges to a but for which there exists an $\varepsilon > 0$ such that, for all n, $|f(x_n) - f(a)| \geq \varepsilon$.)

In closing, note that Theorem 11.1 implies that we could have taken either definition (the more standard definition or the preserving convergence definition) as the definition of continuity. Although these two definitions capture the idea behind continuity in different ways, they are completely equivalent and can be used interchangeably.

Exercises

(1) Let f and g be functions, let a, L, and M be real numbers, and suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$.

(a) Let c be any real number. Prove that $\lim_{x \to a} c \cdot f(x) = c \cdot L$.

(b) Prove that $\lim_{x \to a} [f(x) + g(x)] = L + M$.

(c) Deduce from parts (a) and (b) that $\lim_{x \to a} [f(x) - g(x)] = L - M$.

(d) Prove that $\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$.

(e) Prove that if $M \neq 0$, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

(2) Let f be defined as follows:

$$f(x) = \begin{cases} \dfrac{x^7 - 3}{x - \sqrt[7]{3}} & \text{if } x \neq \sqrt[7]{3} \\ 18 & \text{if } x = \sqrt[7]{3} \end{cases}$$

Does $\lim\limits_{x \to \sqrt[7]{3}} f(x)$ exist? Is f continuous at $\sqrt[7]{3}$? Prove your answers.

(3) Let $f : D \to \mathbb{R}$ be a function. A point $a \in D$ is said to be an *isolated point* of D provided that there exists some $\delta > 0$ such that $(a - \delta, a + \delta) \cap D = \{a\}$. A point $a \in D$ is said to be an *accumulation point* of D provided that there exists some sequence of elements of D, none of which is equal to a, that converges to a.

(a) Prove that a is an isolated point if and only if a is not an accumulation point.

(b) Prove that if a is an accumulation point of D and $\lim\limits_{x \to a} f(x)$ exists, then the limit is unique. In other words, prove that if a is an accumulation point of D and $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} f(x) = M$, then $L = M$.

(c) Prove that if a is an isolated point of D, then for every real number L, $\lim\limits_{x \to a} f(x) = L$.

(d) Prove that if a is an isolated point of D, then f is continuous at a.

(e) Reflecting on your work in parts (a) – (d), explain why some texts require a to be an accumulation point in the definition of $\lim\limits_{x \to a} f(x)$.

(4) Write both an informal and a formal definition of each of the following different types of limits. Assume that a and L are real numbers.

(a) $\lim\limits_{x \to a} f(x) = \infty$

(b) $\lim\limits_{x \to a} f(x) = -\infty$

(c) $\lim\limits_{x \to \infty} f(x) = L$

(d) $\lim\limits_{x \to \infty} f(x) = \infty$

(5) Let a and L be a real numbers, and let f, g, and h be functions such that

- $\lim_{x \to a} f(x) = \infty$
- $\lim_{x \to a} g(x) = -\infty$
- $\lim_{x \to a} h(x) = L$

Using the formal versions of the definitions from Exercise 4, find all possible values of each of the following limits. Prove each of your answers.

(a) $\lim_{x \to a} [f(x) + g(x)]$

(b) $\lim_{x \to a} [f(x) - g(x)]$

(c) $\lim_{x \to a} [f(x) + h(x)]$

(d) $\lim_{x \to a} [f(x) \cdot g(x)]$

(e) $\lim_{x \to a} [f(x) \cdot h(x)]$

(f) $\lim_{x \to a} \frac{f(x)}{g(x)}$

(6) Prove that $f(x) = x^n$ is continuous for every positive integer n. (Hint: Use the Binomial Theorem.)

(7) Prove that f is continuous at a if and only if a is in the domain of f and $\lim_{h \to 0} [f(a + h) - f(a)] = 0$.

(8) For each of the parts below, assume that the given function is continuous at $x = 0$. Use this fact to prove that the function is continuous at every real number. (Hint: Use Exercise 7.)

(a) $f(x) = a^x$, where $a > 0$

(b) $g(x) = \sin(x)$

(c) $h(x) = \cos(x)$

(9) Use the geometry of the unit circle to prove that $\sin(x)$ is continuous at $x = 0$.

(10) Suppose f and g are both continuous at a.

(a) Prove that $f + g$ and $f - g$ are continuous at a.

(b) Prove that fg is continuous at a.

(c) Prove that if f/g is continuous at a if and only if $g(a) \neq 0$.

(11) Prove that if f and g are both continuous, then $f \circ g$ (the composite function defined by $(f \circ g)(x) = f(g(x))$) is continuous at every point a that belongs to the domain of g and for which $g(a)$ belongs to the domain of f.

(12) A function f is said to be *uniformly continuous* on a subset S of the domain of f provided that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in S$,

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$

A function f is said to be *Lipschitz continuous* (and yes, it is pronounced how it is spelled) on a subset S of the domain of f provided that there exists a constant K (called the *Lipschitz constant*) such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in S$.

Use these definitions to prove or disprove each of the following implications. Then summarize your work by stating a theorem that relates continuity, uniform continuity, and Lipschitz continuity. (Hint: You can save yourself some work if you complete the various implications in just the right order.)

(a) If f is continuous on S, then f is uniformly continuous on S.

(b) If f is continuous on S, then f is Lipschitz continuous on S.

(c) If f is uniformly continuous on S, then f is continuous on S.

(d) If f is uniformly continuous on S, then f is Lipschitz continuous on S.

(e) If f is Lipschitz continuous on S, then f is continuous on S.

(f) If f is Lipschitz continuous on S, then f is uniformly continuous on S.

Activity 12

Two Important Theorems about Continuous Functions

Focus Questions

- What is a closed set?
- What does the Extreme Value Theorem say about continuous functions defined on closed and bounded sets, and what are the main ideas behind the proof of the Extreme Value Theorem?
- What is the Intermediate Value Property?
- What does the Intermediate Value Theorem say about continuous functions and the Intermediate Value Property, and what are the main ideas behind the proof of the Intermediate Value Theorem?

Introduction

In this activity, we will continue our investigations of continuous functions by proving two very important and hopefully familiar theorems from calculus. The first of these, the *Extreme Value Theorem*, provides a set of sufficient conditions for a function to attain minimum and maximum values, or extrema. The second, the *Intermediate Value Theorem*, captures a notion that we typically associate with continuous functions: the absence of jumps, skips, gaps, and the like.

The Extreme Value Theorem

In first-semester calculus, you learned (probably within the context of optimization problems) that every continuous function defined on a closed and bounded interval of real numbers attains a minimum and a maximum value on that interval. Here, we will prove a slightly more general version of this result, which is typically called the Extreme Value Theorem. In order to do so, we will first need to define a few terms.

Definition 12.1. Let $f : D \to \mathbb{R}$ be a function.

- If a is an element of D such that $f(a) \leq f(x)$ for all $x \in D$, then f is said to have a *global minimum* at a.
- If a is an element of D such that $f(a) \geq f(x)$ for all $x \in D$, then f is said to have a *global maximum* at a.

When we say that f has a global minimum (or maximum) on some subset S of D, we mean that f, with its domain restricted to S, has a global minimum (or maximum) at some point $a \in S$. Furthermore, we sometimes use the word *extremum* (plural *extrema*) as a generic term for either a minimum or a maximum.

Note that when a function f has a global maximum at a, it must be the case that $f(a)$ is the supremum of the set $f(D)$ defined as follows:

$$f(D) = \{f(x) : x \in D\}$$

Similarly, if f has a global minimum at a, then $f(a)$ is the infimum of $f(D)$.

Question 12.1. For each of the following functions, with the domain of each function restricted to the specified interval, find all of the points at which the function has either a global minimum or a global maximum.

(a) On the interval $[-1, 2]$: $f(x) = \frac{1}{2}x^3 - 2x$

(b) On the interval $[-2, 2]$: $h(x) = \begin{cases} 1 - e^{-x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) On the interval $(-2, 2)$: $k(x) = \sqrt[3]{x}$

(d) On the interval $[-1, 1]$: $m(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ -x^2 & \text{if } x > 0 \end{cases}$

(e) On the interval $(-\infty, \infty)$: $z(x) = \arctan(x)$

(f) On the interval $(-\infty, \infty)$: $\alpha(x) = e^{-x^2}(x^3 - x)$

The next definition we will consider generalizes the notion of a closed interval of real numbers.

Definition 12.2. A set S of real numbers is said to be *closed* provided that S contains the limit of every convergent sequence whose elements belong to S.

Question 12.2. Which of the following sets of real numbers are closed? Use Definition 12.2 to justify each of your answers.

(a) $[-3, 0]$

(b) $(-3, 0]$

(c) $[-3, \infty)$

(d) $(-3, \infty)$

(e) $\{x \in \mathbb{R} : x^2 \leq 2\}$

(f) $\{x \in \mathbb{R} : x^2 < 2\}$

(g) $[-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{2}{3}, \frac{2}{3}] \cup [-\frac{3}{4}, \frac{3}{4}] \cup \cdots$

(h) $S_n = (-\frac{1}{n}, \frac{1}{n}) \cup (1 - \frac{1}{n}, 1 + \frac{1}{n}) \cup (2 - \frac{1}{n}, 2 + \frac{1}{n}) \cup \cdots$

(i) $S_1 \cap S_2 \cap S_3 \cap \cdots$ (where each S_i is as defined in part (h))

As you may have conjectured, every closed interval of real numbers is a closed set. Furthermore, every finite set of real numbers is also closed. We leave the proofs of these facts, as well as several other facts about closed sets, as exercises.

Now that we have defined all of the necessary terms, we are ready to formally state (and prove) the Extreme Value Theorem:

Theorem 12.1 (Extreme Value Theorem)**.** *Let f be a continuous function defined on a nonempty, closed, and bounded set S. Then f has both a global minimum and a global maximum on S.*

Question 12.3. Let f be continuous on S, where S nonempty, closed, and bounded. Follow the steps below to prove the Extreme Value Theorem.

(a) Prove that $f(S)$ is bounded. (Hint: Suppose $f(S)$ is not bounded. Use this fact to construct a convergent sequence $\boldsymbol{x}$ whose elements belong to S and for which $f(\boldsymbol{x})$ does not converge.)

(b) Prove that $f(S)$ is closed.

(c) Use parts (a) and (b) to argue that $f(S)$ has both an infimum ℓ and a supremum m, and that both ℓ and m belong to $f(S)$. (Hint: Construct sequences that converge to ℓ and m.)

(d) Use definition of $f(S)$ to argue that there exist points a, $b \in S$ such that $f(a) = \ell$ and $f(b) = m$

(e) Use part (d), along with the definition of infimum and supremum, to argue that f has a global minimum at a and a global maximum at b.

The Intermediate Value Theorem

Another theorem you most likely studied in first-semester calculus is the Intermediate Value Theorem. This theorem captures the very intuitive idea that the graph of any continuous function can be drawn without lifting one's pencil. In other words, the graphs of continuous functions do not contain any jumps, skips, or gaps. To make this idea more precise, we first define the following property:

Definition 12.3. Let $f : D \to \mathbb{R}$ be a function. Then f is said to satisfy the *Intermediate Value Property* provided that for all a, $b \in D$ with $a < b$ and all k between $f(a)$ and $f(b)$, there exists some number $c \in (a, b)$ such that $f(c) = k$.

Question 12.4. Which of the functions from Question 12.1 satisfy the Intermediate Value Property, and which do not? Give a convincing argument to justify each of your answers.

With our formal definition of the Intermediate Value Property, the Intermediate Value Theorem can then be stated concisely as follows:

Theorem 12.2 (Intermediate Value Theorem[1]). *Let f be a continuous function whose domain is a closed interval of real numbers. Then f satisfies the Intermediate Value Property.*

[1]Incidentally, the Intermediate Value Theorem is often attributed to our friend Bernard Bolzano, who first stated and proved the theorem.

Question 12.5. Prove the Intermediate Value Theorem. (Hint: Let k be a number between $f(a)$ and $f(b)$, and define the set $S = \{x \in [a, b] : f(x) < k\}$. Let $c = \sup(S)$ and argue by contradiction that $f(c) = k$.)

Question 12.6. Does the converse of the Intermediate Value Theorem hold? In other words, if f is a function whose domain is a closed interval of real numbers, and f satisfies the Intermediate Value Property, must f be continuous? Give a proof or counterexample to justify your answer.

Exercises

(1) Prove that every closed interval (that is, every interval that contains its endpoints) is a closed set.

(2) Prove that every set that contains only isolated points is closed. Deduce that every finite set is closed.

(3) (a) Prove that the intersection of any (possibly infinite) collection of closed sets is also closed.

(b) Prove that the union of any finite collection of closed sets is also closed.

(4) Let S be a set of real numbers. A point $p \in S$ is said to be an *interior point* of S provided that there exists a $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq S$. The set S is said to be an *open set* if every element of S is an interior point.

(a) Prove that a set S is open if and only if its complement is closed.

(b) Prove that the union of any (possibly infinite) collection of open sets is also open.

(c) Prove that the intersection of any finite collection of open sets is also open.

(5) Using the definition of *open set* from Exercise 4, prove or disprove each of the following statements:

(a) Every set of real numbers is either open or closed.

(b) A set of real numbers cannot be both open and closed.

(6) Use the Intermediate Value Theorem to prove the *Brouwer Fixed Point Theorem*: If $f : [a, b] \to S \subseteq [a, b]$ is a continuous function, then there exists at least one $x \in [a, b]$ such that $f(x) = x$. (This value of x is called a *fixed point*.)

(7) Use the Intermediate Value Theorem to prove that, at any point in time, there are two *antipodal points* on the earth's surface (that is, points that are on opposite ends of a diameter of the earth) at which the temperature is exactly the same.

(8) Solve the inequality $x^3 - 2x^2 - 5x \geq -6$, identifying each point in your solution at which you used the Intermediate Value Theorem.

(9) Does the Intermediate Value Theorem hold if domain of f is assumed to be a closed set, but not necessarily a closed interval, of real numbers? Give a proof or counterexample to justify your answer.

Activity 13

Derivatives and the Mean Value Theorem

Focus Questions

- What is the precise definition of the derivative, and what does it mean for a function to be differentiable?
- What does the Mean Value Theorem say about the relationship between average and instantaneous rates of change?
- What are the main ideas behind the proof of the Mean Value Theorem?

Introduction

In this activity, we will investigate one of the central objects of study in calculus: the derivative. However, rather than revisiting all of the ideas about the derivative that are typically studied in a first-semester calculus course, we will instead focus on one of the most important theorems about derivatives, the Mean Value Theorem.

Defining the Derivative

You should recall the following definition from your first-semester calculus course:

Definition 13.1. Let $f : D \to \mathbb{R}$ be a function and let $a \in D$. The *derivative* of f at a, denoted $f'(a)$, is defined to be the following limit, provided that it exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit does exist, then we say that f is *differentiable* at a. Otherwise, we say that f is *non-differentiable* at a.

Question 13.1. Discuss Definition 13.1 with one or two of your classmates, using your collective knowledge of first-semester calculus to write a short explanation of the intuition behind the definition of the derivative. Some phrases you might want to include in your definition are: *secant line*, *tangent line*, *average rate of change*, and *instantaneous rate of change*. You may also want to draw a picture to clarify your explanation.

Definition 13.1 defines the derivative of a function at a point, but we can also talk about the *function* f' that maps each point x to the derivative of f at x, $f'(x)$. Note that we sometimes use *Leibniz notation*,

$$\frac{d}{dx}f(x) \text{ or } \frac{df}{dx},$$

to denote this derivative function.

Question 13.2. Use the definition of the derivative to derive the *power rule* for positive integer powers. That is, prove that for every positive integer n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

(Hint: Use the Binomial Theorem.)

Question 13.3. Find a value of a for which the following function is differentiable at a:

$$f(x) = \begin{cases} x^2 - a^2 & \text{if } x \le a \\ 2\sin(ax) & \text{if } x > a \end{cases}$$

Question 13.4. Find the *exact* value of

$$\lim_{h \to 0} \frac{\cos(\pi + h) \cdot 3^{\pi+h} - \cos(\pi) \cdot 3^{\pi}}{h}.$$

(Hint: Use Definition 13.1.)

The Mean Value Theorem

One of the most important theorems about derivatives is the Mean Value Theorem, which establishes a relationship between average and instantaneous rates of change. The formal statement of the Mean Value Theorem is given below:

Theorem 13.1 (Mean Value Theorem)**.** *Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Question 13.5.

(a) Draw a picture to illustrate the Mean Value Theorem. Clearly explain your picture using the following phrases: *secant line*, *tangent line*, *average rate of change*, and *instantaneous rate of change*. (Sound familiar?)

(b) Demonstrate how each of the hypotheses of continuity and differentiability are essential to the statement of the Mean Value Theorem. That is, show that the conclusion of the Mean Value Theorem can fail to hold if either of these hypotheses are not satisfied.

The Mean Value Theorem can actually be proved using the Extreme Value Theorem. However, in order to do so, we will first need to prove a lemma about derivatives and local extrema. We'll begin with the following definition:

Definition 13.2. Let $f : D \to \mathbb{R}$ be a function, and let $a \in D$. Then:

- f is said to have a *local minimum* at a if there exists some $\delta > 0$ such that for all $x \in D \cap (x - \delta, x + \delta)$, $f(a) \leq f(x)$.
- f is said to have a *local maximum* at a if there exists some $\delta > 0$ such that for all $x \in D \cap (x - \delta, x + \delta)$, $f(a) \geq f(x)$.

As we did when we talked about global minima and maxima, we will sometimes use the phrase *local extremum* as a generic term for either a local minimum or a local maximum.

Question 13.6. Find all of the local extrema of the function $f(x) = |xe^{\sin(x)}|$ on the interval $[-5, 5]$. Which, if any, of these local extrema are global extrema?

The following lemma should be familiar to you from your first-semester calculus course:

Lemma 13.2. *Let $f : D \to \mathbb{R}$ be a function, and suppose that f has a local extremum at some point $a \in D$. If f is differentiable at a, then $f'(a) = 0$.*

Question 13.7. Prove Lemma 13.2 in the case that f has a local maximum at a. (Hint: Consider the quantities

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}.$$

Use the fact that f has a local maximum at a to argue that one of these quantities must be nonpositive and one must be nonnegative. Then argue that both quantities must be equal.[1])

With Lemma 13.2 in hand, we are now ready to prove the Mean Value Theorem. We'll begin with a special case known as *Rolle's Theorem*:

Theorem 13.3 (Rolle's Theorem)**.** *Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose also that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Question 13.8. Use Lemma 13.2 and the Extreme Value Theorem to prove Rolle's Theorem. (Hint: Consider two cases based on the location within $[a, b]$ of the global extrema guaranteed by the Extreme Value Theorem.)

Question 13.9. Use Rolle's Theorem to prove the Mean Value Theorem. (Hint: For any function f that satisfies the hypotheses of the Mean Value Theorem, define a new function g that satisfies the hypotheses of Rolle's Theorem. If you define g in just the right way, the conclusion of the Mean Value Theorem (applied to f) will follow directly from the conclusion of Rolle's Theorem (applied to g).)

Exercises

(1) Let f and g be functions. Use the definition of the derivative to prove that if both f and g are differentiable at a point a, then:

   (a) $f + g$ is differentiable at a, and $(f + g)'(a) = f'(a) + g'(a)$.

[1] The so-called *sided limits* involved in this question can be defined precisely as follows:

- $\lim_{h\to 0^-} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$,
$$x \in (a-\delta, a) \text{ implies } |f(x) - L| < \varepsilon.$$
- $\lim_{h\to 0^+} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$,
$$x \in (a, a+\delta) \text{ implies } |f(x) - L| < \varepsilon.$$

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(b) fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. (Hint: At some point, you may want to add and subtract the same quantity.)

(2) You may recall the following theorem from your first-semester calculus course:

Theorem 13.4. *If a function f is differentiable at a point a, then f is also continuous at a.*

(a) Prove Theorem 13.4. (Hint: Reason by contradiction, using the fact that f is continuous at a if and only if a is in the domain of f and $\lim_{h \to 0} [f(a+h) - f(a)] = 0$.)

(b) Is the converse of Theorem 13.4 true? Give a proof or counterexample to justify your answer.

(3) Let f be a differentiable function.

(a) Prove that if $f'(x) > 0$ for all x in some interval (a, b), then f is increasing[2] on (a, b).

(b) Is the converse of the statement from part (a) true? If so, prove it. Otherwise, give a counterexample and prove a closely related true statement.

(4) Prove that if $f'(x) = 0$ for all x in some interval (a, b), then f is constant on (a, b).

(5) Is the following statement true or false?

> Let $f : D \to \mathbb{R}$ be a function, let $a \in D$, and suppose there exists $\delta > 0$ such that f is increasing on $(a - \delta, a)$ and decreasing on $(a, a + \delta)$. Then f has a local maximum at a.

If it is true, prove it. Otherwise, give a counterexample and prove a closely related true statement.

(6) Prove the following theorem, known as Cauchy's Mean Value Theorem:

[2] A function f is said to be *increasing* on an interval (a, b) provided that for all $x, y \in (a, b)$, $x < y$ implies $f(x) < f(y)$. A function f is said to be *decreasing* on (a, b) provided that for all $x, y \in (a, b)$, $x < y$ implies $f(x) > f(y)$. Similar definitions can be made for the terms *never decreasing* (or *non-decreasing*) and *never increasing* (or *non-increasing*) by replacing the inequalities $f(x) < f(y)$ and $f(x) > f(y)$ with $f(x) \leq f(y)$ and $f(x) \geq f(y)$, respectively.

Theorem 13.5 (Cauchy's Mean Value Theorem)**.** *Let f and g be functions, and suppose that both f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

(Hint: As with the proof of the Mean Value Theorem, apply Rolle's Theorem to a conveniently defined function.)

(7) Recall that a function f is said to be *Lipschitz continuous* on a subset S of the domain of f provided that there exists a constant K (called the *Lipschitz constant*) such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in S$. (See Exercise 12 from Activity 11.) Suppose that a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Prove that f is Lipschitz continuous on $[a, b]$ if and only if f' is bounded on (a, b).

Activity 14

The Riemann Integral

Focus Questions

- What is the precise definition of the Reimann integral, and what does it mean for a function to be Riemann integrable?
- What are some sufficient conditions for Riemann integrability?
- What does the Fundamental Theorem of Calculus say, and what are the main ideas behind its proof?

Introduction

In your first-semester calculus course, you probably ended the semester by studying the Riemann integral. In this activity, we will consider the Riemann integral once again, but this time from a more general, theoretical perspective. By doing so, we'll be able to fill in some of the gaps that often remain after an introductory treatment of integration.

Defining the Integral

In order to define the Riemann integral, we will first need to consider some preliminary definitions and results.

Definition 14.1. Let $[a, b]$ be a closed interval of real numbers. A *partition* of $[a, b]$ is a finite set $\{x_k\}_{k=0}^{n} = \{x_0, x_1, x_2, \ldots, x_n\}$ of points in $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b.$$

For any given partition of $[a, b]$, we typically use the notation Δx_k to denote the difference between successive points in the partition, so that $\Delta x_k = x_k - x_{k-1}$. Furthermore, if Q is a partition of $[a, b]$ and $P \subseteq Q$, then we say that Q is a *refinement* of P.

Definition 14.2. Let f be a bounded function whose domain contains $[a, b]$, and let $P = \{x_k\}_{k=0}^{n}$ be a partition of $[a, b]$. For each k, define

$$m_k = \inf(\{f(x) : x \in [x_{k-1}, x_k]\})$$

and

$$M_k = \sup(\{f(x) : x \in [x_{k-1}, x_k]\}).$$

A *Riemann sum* is any sum of the form

$$\sum_{k=1}^{n} f(c_k)\Delta x_k,$$

where $c_k \in [x_{k-1}, x_k]$ for each k. The *lower sum* of f with respect to P is the sum

$$L_P(f) = \sum_{k=1}^{n} m_k \Delta x_k.$$

The *upper sum* of f with respect to P is the sum

$$U_P(f) = \sum_{k=1}^{n} M_k \Delta x_k.$$

Question 14.1. For the partition $P = \{0, 0.2, 0.4, \ldots, 1.0\}$ of $[0, 1]$ and for each of the following functions, calculate the lower sum $L_P(f)$, the upper sum $U_P(f)$, and a Riemann sum that is neither an upper sum nor a lower sum.

(a) $f(x) = 1$

(b) $f(x) = x^2$

(c) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

(d) $f(x) = \begin{cases} 1 & \text{if } 10x \in \mathbb{Z} \\ x^2 & \text{if } 10x \notin \mathbb{Z} \end{cases}$

Question 14.2. Let f be a function defined on $[a,b]$, and let P be a partition of $[a,b]$.

(a) Prove that if f is bounded, then there exist real numbers m and M such that

$$m(b-a) \leq L_P(f) \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq U_P(f) \leq M(b-a),$$

where $\sum_{k=1}^{n} f(c_k)\Delta x_k$ is any Riemann sum for f (with respect to the partition P).

(b) Prove that if Q is a refinement of P, then

$$L_P(f) \leq L_Q(f) \leq U_Q(f) \leq U_P(f).$$

(Hint: First consider the case where Q is formed by adding one additional point to P.)

(c) Prove that for any partition Q of $[a,b]$, $L_Q(f) \leq U_P(f)$. (Hint: Use part (b), and consider the partition $P \cup Q$.)

Now that we understand partitions and Riemann sums, we can formally define the Riemann integral as follows:

Definition 14.3. Let f be a bounded function defined on $[a,b]$. Then the *lower integral* of f on $[a,b]$ is denoted and defined as follows:

$$\underline{\int_a^b} f(x)\,dx = \sup\{L_P(f) : P \text{ is a partition of } [a,b]\}$$

The *upper integral* of f on $[a,b]$ is denoted and defined as follows:

$$\overline{\int_a^b} f(x)\,dx = \inf\{U_P(f) : P \text{ is a partition of } [a,b]\}$$

If the value of the lower and upper integrals of f are equal, then f is said to be *Riemann integrable* on $[a,b]$, and the common value of the lower and upper integrals is called the *Riemann integral* of f on $[a,b]$, denoted

$$\int_a^b f(x)\,dx.$$

Question 14.3. How is the definition of the integral given above similar to the one that you learned in your first-semester calculus course? How is it different? Be specific and precise.

Question 14.4. Which of the functions from Question 14.1 are Riemann integrable? For those that are, find the value of the Riemann integral. For those that are not, show that the corresponding upper and lower integrals are not equal.

Incidentally, the Riemann integral is named after Georg Friedrich Bernhard Riemann, a German mathematician whose research on the geometry of space influenced much of the development of modern theoretical physics. Riemann was one of the first to formalize the notion of the integral, which was introduced by Cauchy in the early 1800s. Riemann's formulation of the integral is the one most commonly taught in college calculus courses. There are, however, more general versions due to mathematicians such as Lebesgue, Stiltjes, and Darboux.

Georg Friedrich Bernhard Riemann
(1826 - 1866)

Conditions for Riemann Integrability

Many interesting questions surround the notion of integrability. The first question that we will consider concerns finding sufficient conditions for Riemann integrability (that is, conditions that, if satisfied, will guarantee that the function in question is Riemann integrable). Though the conditions themselves are relatively straightforward and not terribly surprising, their proofs can be slightly more challenging, partly because of the complexity of the definition of the Riemann integral.

The two main results we will prove in this section are the following:

Theorem 14.1. *If a function f is monotone (that is, either never increasing or never decreasing) and defined for all $x \in [a, b]$, then f is Riemann integrable on* $[a, b]$.

Theorem 14.2. *If a function f is a continuous on $[a, b]$, then f is Riemann integrable on* $[a, b]$.

In order to prove these results, we will need the following lemmas:

Lemma 14.3. *Let f be a function that is defined and bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P such that $U_P(f) - L_P(f) < \varepsilon$.*

Question 14.5. Prove the "if" direction of Lemma 14.3. That is, prove that if for every $\varepsilon > 0$, there exists a partition P such that $U_P(f) - L_P(f) < \varepsilon$, then f is Riemann integrable on $[a, b]$.

$$\left(\text{Hint: Note that } L_P(f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U_P(f).\right)$$

If you completed Exercise 12 from Activity 11, then you should recognize the next lemma. Stated concisely, it says that every continuous function defined on a closed and bounded interval is uniformly continuous.

Lemma 14.4. *Let f be a function that is continuous on $[a, b]$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, $y \in [a, b]$,*

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$

Question 14.6. Prove Lemma 14.4. (Hint: Use a proof by contradiction to construct two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ such that, for all n, $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Apply the Bolzano-Weierstrass Theorem and the "preserving convergence" property of continuity to these sequences in order to arrive at the desired contradiction.)

We are now ready to prove Theorems 14.1 and 14.2. The next two questions suggest strategies for each.

Question 14.7. Suppose f is never decreasing and defined for all $x \in [a, b]$.

(a) Using the assumption that f is never decreasing on $[a, b]$, explain why f must be bounded on $[a, b]$.

(b) Use part (a) to argue that for every $\varepsilon > 0$, there must exist a real number $K > 0$ such that
$$f(b) - f(a) < \frac{\varepsilon}{K}.$$

(c) Choose a partition $P = \{x_k\}_{k=0}^{n}$ of $[a, b]$ such that $\Delta x_k < K$ for all k. Explain why such a partition must exist.

(d) Argue that for the partition P you chose in part (c), $U_P(f) - L_P(f) < \varepsilon$. (Hint: Expand $U_P(f) - L_P(f)$, keeping in mind that f is never decreasing. Also note that $K \cdot \dfrac{\varepsilon}{K} = \varepsilon$.)

(e) Explain how your work in parts (a)–(d) establishes Theorem 14.1 for never decreasing functions.

(f) Explain how your argument from parts (a)–(e) could be modified to prove the same result for never increasing functions.

Question 14.8. Suppose f is continuous on $[a, b]$.

(a) Argue that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

(b) Choose a partition $P = \{x_k\}_{k=0}^{n}$ of $[a, b]$ such that $\Delta_k < \delta$ for all k. Explain why such a partition must exist.

(c) Explain why for each $k \in \{1, 2, \ldots n\}$, there must exist s_k and t_k in $[x_{k-1}, x_k]$ such that $m_k = f(s_k)$ and $M_k = f(t_k)$. (Hint: Apply a theorem we proved in a previous activity.)

(d) Use parts (a)–(c) to argue that for each $k \in \{1, 2, \ldots n\}$,

$$f(t_k) - f(s_k) < \frac{\varepsilon}{b - a}.$$

(e) Use part (d) to argue that $U_P(f) - L_P(f) < \varepsilon$.

(f) Explain how your work in parts (a)–(e) establishes Theorem 14.2.

The Fundamental Theorem of Calculus

The results we have considered thus far have given us conditions under which the Riemann integral is guaranteed to exist. These results, however, have not told us anything about how to actually calculate the value of the Riemann integral of a function. In fact, the formal definition of the integral is so complex that it is almost always of no practical value when it comes to solving problems that involve actually evaluating integrals.

If you remember nothing else from your first-semester calculus class, you should remember the following theorem, which beautifully ties together differentiation and integration in a way that makes evaluation of Riemann integrals a much more practical task. We will end our investigations of integration in this activity by proving this very important theorem.

Theorem 14.5 (Fundamental Theorem of Calculus)**.** *Let f be a function that is continuous on $[a, b]$, and let $F : [a, b] \to \mathbb{R}$ be any function such that $F'(x) = f(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Question 14.9. Suppose f is continuous on $[a, b]$, and let $P = \{x_k\}_{k=0}^n$ be any partition of $[a, b]$.

(a) Apply the Mean Value Theorem to the function F on each subinterval $[x_{k-1}, x_k]$ of P. Begin your conclusion as follows: *There exists $c_k \in (x_{k-1}, x_k)$ such that ...*

(b) Let S be the Riemann sum defined by

$$S = \sum_{k=1}^{n} f(c_k)\Delta_k,$$

where each c_k is as specified in the conclusion of the Mean Value Theorem from part (a). Argue that $S = F(b) - F(a)$.

(c) Use part (b) to argue that for any lower and upper sums, say $L_P(f)$ and $U_P(f)$, respectively,

$$L_P(f) \le F(b) - F(a) \le U_P(f).$$

(d) Use part (c) and Theorem 14.2 to finish the proof of the Fundamental Theorem of Calculus.

Exercises

(1) Are the sufficient conditions from Theorems 14.1 and 14.2 also necessary conditions? That is, if a function is Riemann integrable on $[a, b]$, must f be monotone on $[a, b]$? Must f be continuous on $[a, b]$? Give proofs or counterexamples to justify your answers.

(2) Prove that if f is continuous and nonnegative on $[a, b]$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

(3) Let f and g be Riemann integrable on $[a, b]$. Prove that if f and g differ only on a set of isolated points (see Exercise 3 from Activity 11), then

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

(4) Prove that if f and g are Riemann integrable on $[a, b]$, then $f + g$ and $f - g$ are also Riemann integrable on $[a, b]$, and

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

(5) Prove that if f is Riemann integrable on $[a, b]$ and c is any constant, then

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

(6) Prove that for any real numbers a, b, and c,

$$\int_a^b c\,dx = c(b - a).$$

(7) Prove that if f is Riemann integrable on $[a, b]$ and $c \in [a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(8) (a) Prove that if f is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$, and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

(b) If $|f|$ is Riemann integrable on $[a, b]$, must f also be? Give a proof or counterexample to justify your answer.

(9) Let f be continuous on $[a, b]$, and let F be the function defined by

$$F(x) = \int_a^x f(t)\,dt.$$

(a) Explain why F is defined for all $x \in [a, b]$.

(b) Prove that F is an antiderivative of f. That is, prove that F is differentiable and that $F'(x) = f(x)$ for all $x \in [a, b]$. (Hint: Use the properties from Exercises 4 – 8 to show that the quantity

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right|$$

can be made less than an arbitrary $\varepsilon > 0$ by choosing $|h| < \delta$, where δ is given by Lemma 14.4.)

Note that the result you proved in part (b) is a very important theorem called the *Second Fundamental Theorem of Calculus* or the *Construction Theorem* (for antiderivatives). It establishes that every continuous function has an antiderivative, even if that antiderivative may not be able to be expressed in simple terms.

Appendix A

A Menu of Sequences

(1) Let $\boldsymbol{s}$ be the sequence defined by applying Newton's method to

$$p(x) = \frac{4}{3}\,x^2 - 4,$$

with starting point $x_0 = \frac{7}{4}$.

(2) Let $g(x) = \frac{1}{2}\,\cos(\pi x)$, and define the sequence $\boldsymbol{s}$ by

$$s_1 = g(1),\ s_2 = g'(1),\ s_3 = g''(1), \ldots,\ s_n = g^{(n-1)}(1), \ldots.$$

(3) Let $f(x) = 3\,\sin(\alpha x)$, and define the sequence $\boldsymbol{s}$ by

$$s_1 = f(1),\ s_2 = f'(1),\ s_3 = f''(1), \ldots,\ s_n = f^{(n-1)}(1), \ldots.$$

where:

(a) $\alpha = \frac{\pi}{2}$ (b) $\alpha = \frac{\pi}{3}$ (c) $\alpha = \frac{\pi}{4}$

(4) Let $h(x) = \frac{1}{2}\,x + \frac{3}{2}\,x^{-1}$. Define $\boldsymbol{s}$ by

$$\begin{aligned} s_1 &= 2, \\ s_{n+1} &= h(s_n), \quad \text{for all } n \geq 1. \end{aligned}$$

(5) Define $\boldsymbol{s}$ by $s_n = \cos(n)$ for each positive integer n.

(6) Let $\boldsymbol{s}$ be the sequence defined by applying Newton's method to

$$q(x) = \frac{5}{2}\,x^2 + 5,$$

with starting point $x_0 = 2$.

(7) Define s by

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}.$$

for each positive integer n.

(8) Define s by

$$s_n = \sum_{k=1}^{n} \frac{1}{k}.$$

for each positive integer n.

(9) Define s by

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

for each positive integer n.

(10) Define s by

$$\begin{aligned} s_1 &= 6.1, \\ s_2 &= 4.6, \\ s_{n+2} &= \frac{7}{4} s_{n+1} - \frac{3}{4} s_n, \text{ for all } n \geq 1. \end{aligned}$$

(11) Define s by

$$s_n = 1 - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2k^2 + k}$$

for each positive integer n.

Index